MATHS Xtra

Unit 24

Addition of waves,
periodic motion and modelling

Objectives

On completion of this unit you should understand:

1. The frequency of sine and cosine functions.

2. The shapes of the $\sin^2 x$ and $\cos^2 x$ curves.

3. The amplitude, period, frequency and phase angle of waveforms.

4. The resultant formula for the addition of two sine waveforms.

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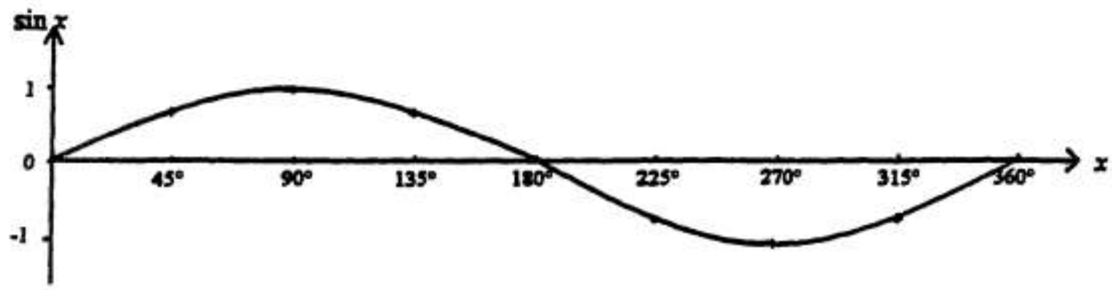
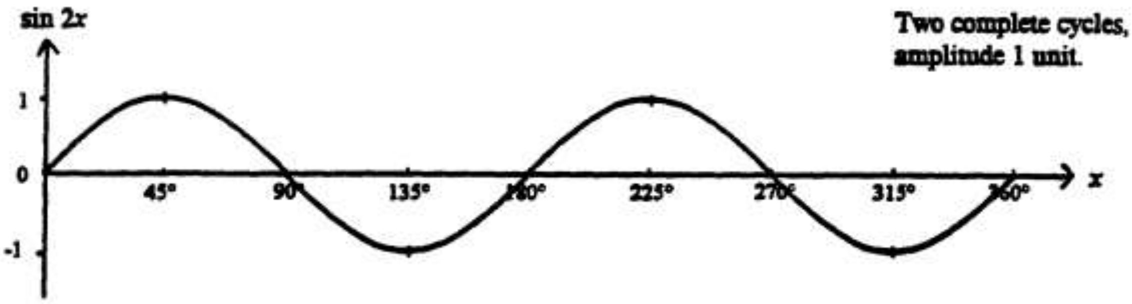
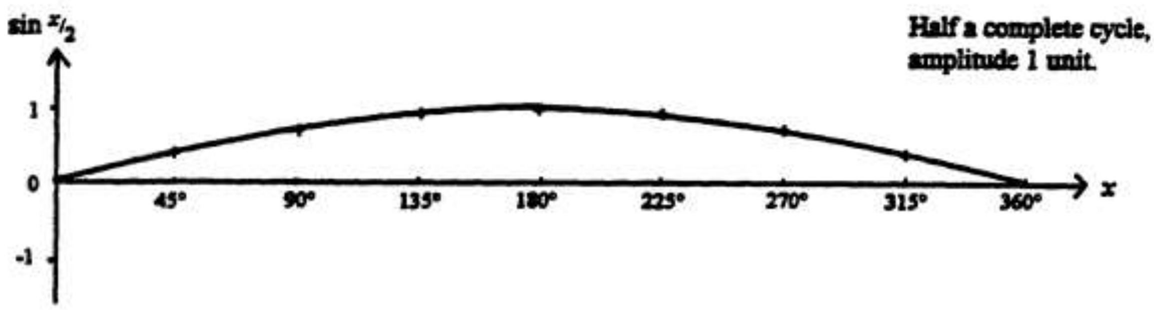
Amplitude

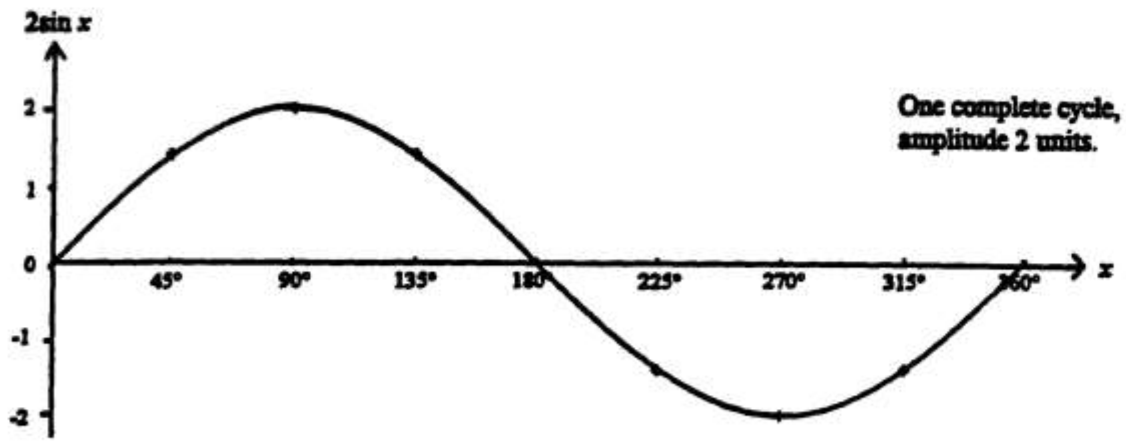
Below we have one complete cycle of the graph of sin x, which you have seen before. A complete cycle is the section of the waveform, which shows its complete shape. You should be able to see from the graph that the maximum value of sin x is +1 and the minimum value is -1. The amplitude of the curve is 1 unit.



We can use a calculator and obtain the curves for sin $2x$, sin $^{x}/_{2}$, and 2sin x for values of x from 0° to 360°. These are shown below.







Try this exercise.

Exercise A
Draw the curves for cos x, cos $2x$, and cos $x/2$, from 0° to 360°. Write the number of cycles and the amplitude in each case.

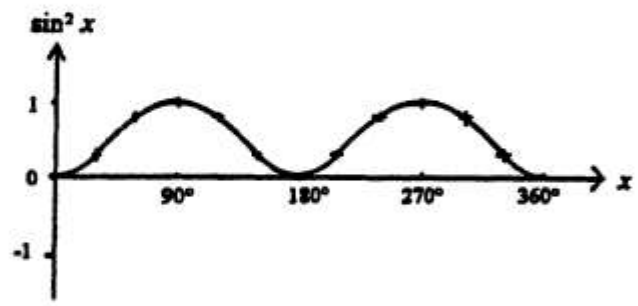
Check your answers with those at the end of the unit, then study this example.

Example 1
Draw the graph for $\sin^2 x$ for values of x from 0° to 360°.

Taking values from a calculator, we obtain the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin^2 x$	0	0.25	0.75	1	0.75	0.25	0	0.25	0.75	1	0.75	0.25	0

The graph shown below is never negative.



Try this exercise.

Exercise B
Make a table of values from 0° to 360° as shown in Example 1. Use it to draw the graph for $\cos^2 x$ for values of x from 0° to 360°.

Check your table of values with the one given at the end of the unit.

Angular and time scales

In the diagram below, the circle, centre O, has a radius R. P is a point on the circumference of the circle.

If OP rotates at a uniform angular velocity ω radians per second in an anticlockwise direction, then,

angle turned through = angular velocity x time taken

so, after t seconds,

angle turned through = ωt radians.

From the triangle OPQ,

$$\sin POQ = \frac{PQ}{OP}$$

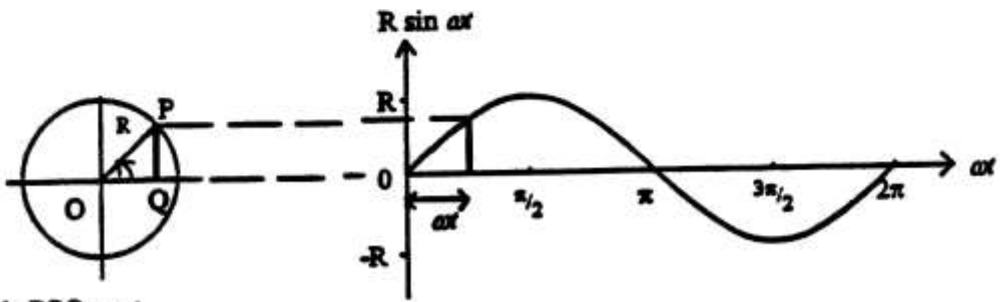
so,

$$PQ = OP \sin POQ$$

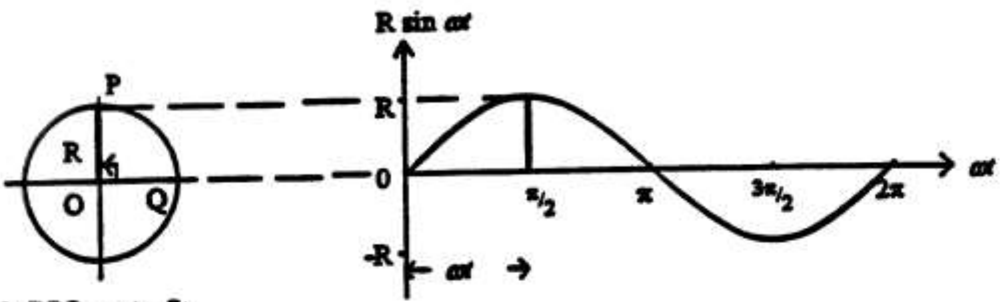
$$PQ = R\sin \omega t.$$

As P rotates the length of PQ is measured at every point around the circle and a graph can be drawn to show how PQ varies with the angle ωt. The length of PQ is plotted on the vertical axis and the angle ωt is plotted on the horizontal axis.

The sine wave representing R$\sin \omega t$ is obtained. You should note from the graph that the amplitude of the graph is the value R.



Angle POQ = ωt



Angle POQ = $\omega t = \pi/2$

One complete cycle is completed whilst the radius OP turns through 360° or 2π radians.

Time period and frequency

The time period is the time taken, in seconds, for one complete cycle.

$$\text{Time taken} = \frac{\text{angle turned through}}{\text{angular velocity}}$$

so,

$$\textbf{time period} = \frac{2\pi}{\omega} \text{ seconds.}$$

The frequency is the number of cycles per second and is measured in hertz (Hz).

∴ One cycle takes $\frac{2\pi}{\omega}$ seconds, so in one second there are $\frac{\omega}{2\pi}$ cycles.

$$\textbf{Frequency} = \frac{\omega}{2\pi} \text{ hertz.}$$

From this it can be seen that,

$$\textbf{frequency} = \frac{1}{\textbf{time period}}$$

and,

$$\textbf{time period} = \frac{1}{\textbf{frequency}}.$$

Study this example.

Example 2

Using the information above, find the time period for the graphs of sin t, sin $2t$ and sin $^t/_2$.

From the previous page, the graph of Rsin ωt completed one compete cycle in 2π radians.

Time = angular displacement ÷ angular velocity

For one complete cycle the angular displacement is 2π and the angular velocity is ω,

so, the time period $= \frac{2\pi}{\omega}$ seconds.

Using the same logic,

the time period for sin t is $2\pi \div 1 = 2\pi$ or 6.28 seconds,
the time period for sin $2t$ is $2\pi \div 2 = \pi$ or 3.14 seconds,
the time period for sin $^t/_2$ is $2\pi \div {}^1/_2 = 4\pi$ or 12.56 seconds.

Try this exercise, then check your answers with those at the end of the unit.

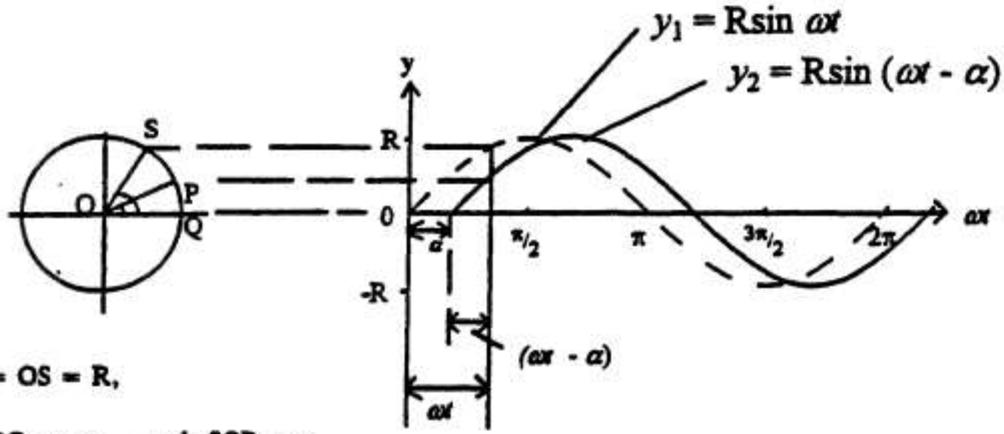
Exercise C

Calculate the time period for each of the following.

1. sin $4t$ 2. sin $5t$ 3. sin $^t/_4$ 4. sin $^t/_8$

Phase angle

The diagram below shows two sine waves. They are identical, but they are displaced from each other by the angle α. α is known as the phase difference or phase angle. OP and OS are known as phasors. In the diagram, OP lags behind OS by angle α. Both curves have the same amplitude, so that R is the same for each.



If, OP = OS = R,
and
angle SOQ = ωt angle SOP = α
then,
angle POQ = $(\omega t - \alpha)$.

On the diagram, the curve for the phasor OS is y_1 and that for the phasor OP is y_2. We can see from the circle on the left that,

$y_1 = R\sin \omega t$ and $y_2 = R\sin(\omega t - \alpha)$.

Note that the graph of $R\sin(\omega t - \alpha)$ is displaced to the **right** of the graph of $R\sin \omega t$ by the angle α.
Similarly the graph of $R\sin(\omega t + \alpha)$ would be displaced to the left of $R\sin \omega t$ by the angle α. You should remember this fact from the Trig Graphs unit.

A graph of $R\sin(\omega t + \pi/2)$ would lead the graph of $R\sin \omega t$ by $\pi/2$.
Similarly a graph of $R\sin(\omega t - \pi/4)$ would lag behind the graph of $R\sin \omega t$ by $\pi/4$.

Study the example on the next page.

Example 3

Given that an instantaneous current is given by $i_1 = 5 \sin(3t + {}^{\pi}/_{3})$,

a) state the amplitude,
b) find the time period and phase angle,
c) find the frequency,
c) state whether this curve lags or leads that of $i_2 = 5 \sin 3t$.

a) The amplitude is 5amps.

b) The time period is $\frac{2\pi}{3} = 2.09$ seconds.

The phase angle is ${}^{\pi}/_{3}$ radians.

c) The frequency $= \frac{1}{\text{period}}$

$= \frac{3}{2\pi} = 0.477$ Hz. to three significant figures.

d) The curve leads that of i_2.

Try this exercise.

Exercise D

For each of the following currents i_1, state.

a) the amplitude,
b) the period,
c) the frequency and
d) the phase angle, stating whether i_1 is leading or lagging i_2.

Give answers correct to two decimal places, where appropriate.

1. $i_1 = 6 \sin(4t - {}^{\pi}/_{4})$ — $i_2 = 6 \sin 4t$
2. $i_1 = 15 \sin(0.5t + {}^{\pi}/_{6})$ — $i_2 = 15 \sin 0.5t$
3. $i_1 = 1.3 \sin(2\pi t - {}^{\pi}/_{2})$ — $i_2 = 1.3 \sin 2\pi t$
4. $i_1 = 0.8 \sin(5\pi t + {}^{\pi}/_{12})$ — $i_2 = 0.8 \sin 5\pi t$
5. $i_1 = 25 \sin(2t - 0.15)$ — $i_2 = 25 \sin 2t$

Check your answers with those given at the end of this unit.

Combination of sine waves

Study these examples.

Example 4

Two sine waves, i_1 and i_2, are given below.

$$i_1 = 4\sin\theta \qquad \text{and} \qquad i_2 = 4\sin(\theta + 45°)$$

Find the resultant waveform i_r.

We can do this algebraically by adding i_1 and i_2 as follows,

$$i_r = 4\sin\theta + 4\sin(\theta + 45°)$$

We use the double angle formula to expand i_2,

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

so that, $i_r = 4\sin\theta + 4\sin\theta\cos 45° + 4\cos\theta\sin 45°$

but $\sin 45° = \cos 45° = {}^1/_{\sqrt{2}}$

so, $i_r = 4\sin\theta + 4({}^1/_{\sqrt{2}})\sin\theta + 4({}^1/_{\sqrt{2}})\cos\theta$

$$i_r = \{4 + 4({}^1/_{\sqrt{2}})\}\sin\theta + 4({}^1/_{\sqrt{2}})\cos\theta$$

$$i_r = 6.8284271\sin\theta + 2.8284271\cos\theta$$

We can now express this as a single waveform by using the formula

$$R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha.$$

So, $6.8284271\sin\theta + 2.8284271\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha.$

Compare the coefficients of $\sin\theta$ and $\cos\theta$.

$R\cos\alpha = 6.8284271$ ------(1) and $R\sin\alpha = 2.8284271$ ------(2)

so that, if we square and add equations (1) and (2) we have

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 6.8284271^2 + 2.8284271^2$$

$R^2(\cos^2\alpha + \sin^2\alpha) = 54.627417$ but $\cos^2\alpha + \sin^2\alpha = 1.$

So that, $R^2 = 54.627417$

$R = 7.4$ to one decimal place.

Equation (2) divided by (1) gives $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{2.8284271}{6.8284271}$

but $\tan\alpha = {}^{\sin\alpha}/_{\cos\alpha}$

so, $\tan\alpha = \dfrac{2.8284271}{6.8284271}$ $\qquad \alpha = 22.5°$

so, $i_r = 7.4\sin(\theta + 22.5°)$.

i_r is a sine wave with an amplitude of 7.4. It leads the sine wave for $7.4\sin\theta$ by 22.5°.

Example 5

Two sine waves, i_1 and i_2, are given below.

$i_1 = 3\sin\theta$ and $i_2 = 7\sin(\theta - 30°)$

Find the resultant waveform i_r.

We can do this algebraically by adding i_1 and i_2 as follows,

$$i_r = 3\sin\theta + 7\sin(\theta - 30°)$$

We use the double angle formula to expand i_2,

$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

so that,

$$i_r = 3\sin\theta + 7\sin\theta\cos 30° - 7\cos\theta\sin 30°$$
$$i_r = 3\sin\theta + 7(\sqrt{3}/2)\sin\theta - 7(1/2)\cos\theta$$
$$i_r = \{3 + 7(\sqrt{3}/2)\}\sin\theta - 7(1/2)\cos\theta$$
$$i_r = 9.0621778\sin\theta - 3.5\cos\theta$$

We can now express this as a single waveform by using the formula

$$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha.$$

So, $9.0621778\sin\theta - 3.5\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$.

Compare the coefficients of $\sin\theta$ and $\cos\theta$.

$R\cos\alpha = 9.0621778$ ——(1) and $R\sin\alpha = 3.5$ ——(2)

Square and add (1) and (2) as before,

$$R^2(\cos^2\alpha + \sin^2\alpha) = 9.0621778^2 + 3.5^2$$
$$R^2 = 9.0621778^2 + 3.5^2$$

$R = 9.7$ to one decimal place.

Equation (2) ÷ (1) gives $\tan\alpha = \dfrac{3.5}{9.0621778}$ $\alpha = 21.1°$

so, $i_r = 9.7\sin(\theta - 21.1°)$.

i_r is a sine wave with an amplitude of 9.7. It lags behind the $\sin\theta$ curve by 21.1°.

Try this exercise, then check your answers with those at the end of the unit.

Exercise E

Using an algebraic method, find the resultant waveform in each case. Give all answers correct to one decimal place.

1. $5\sin\theta + 6\sin(\theta + 60°)$
2. $2\sin\theta + 3\sin(\theta - 45°)$
3. $4\sin\theta + \sin(\theta + 15°)$

The result of two waveforms may also be found by simply drawing the graphs and adding them together, or by completing a table of values as shown below. *Study this example.*

Example 6

Two sine waves, i_1 and i_2, are given below.

$i_1 = 5\sin\theta$ and $i_2 = 5\sin(\theta + 30°)$

Using a table of values, find the resultant waveform i_r.

We list the values for i_1 and i_2, then add them together to find i_r.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$5\sin\theta$	0	2.5	4.33	5	4.33	2.5	0	-2.5	-4.33	-5	-4.33	-2.5	0
$5\sin\theta + 30°$	2.5	4.33	5	4.33	2.5	0	-2.5	-4.33	-5	-4.33	-2.5	0	2.5
i_r	2.5	6.83	9.33	9.33	6.83	2.5	-2.5	-6.83	-9.33	-9.33	-6.83	-2.5	2.5

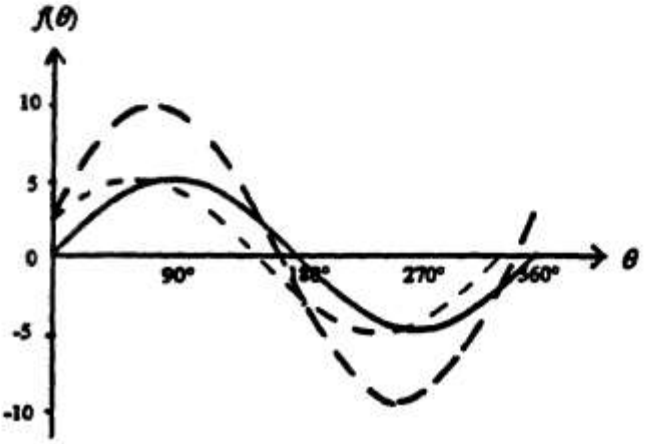
We can see from the table that the turning points for the graph must be half way between 60° and 90° and half way between 240° and 270°. We can find these using a calculator.

Calculator: 5 x 75 sin + 5 x (75 + 30) sin = 9.659258

Calculator: 5 x 255 sin + 5 x (255 + 30) sin = -9.659258

When $\theta = 75°$, $i_r = 9.66$. When $\theta = 255°$, $i_r = -9.66$.

The graphs are sketched below.



The values to be plotted for the resultant curve can be found by adding the height of the curve $5\sin(\theta + 30°)$ to that of $5\sin\theta$ for each value of θ.

If these are drawn on graph paper it can be done by simply counting squares and adding them together. For example count the number of squares to find the height of $5\sin(\theta + 30°)$ at $\theta = 30°$. Do the same for $5\sin\theta$ at $\theta = 30°$ and add them together so that the new point can be plotted.

Try this last exercise.

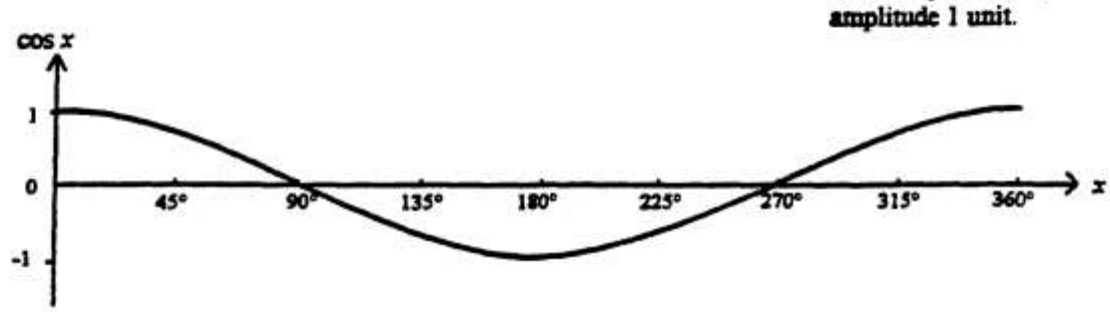
Exercise F

Repeat the questions in Exercise E but this time find the equation of the resultant wave form by adding together the graphs of each curve.

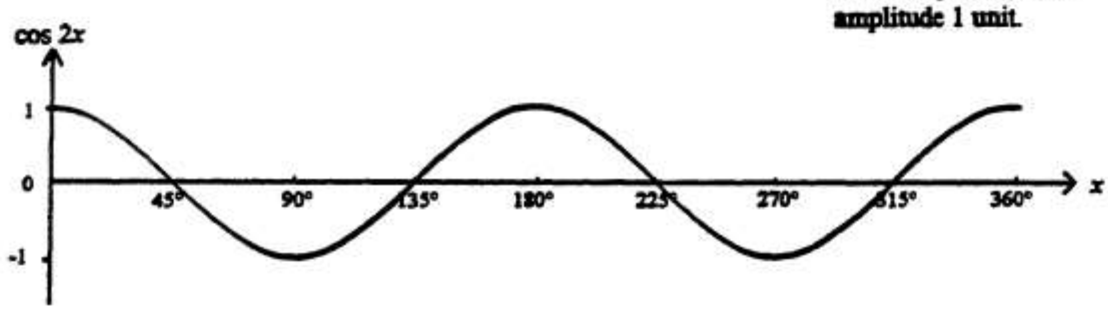
Answers

Exercise A

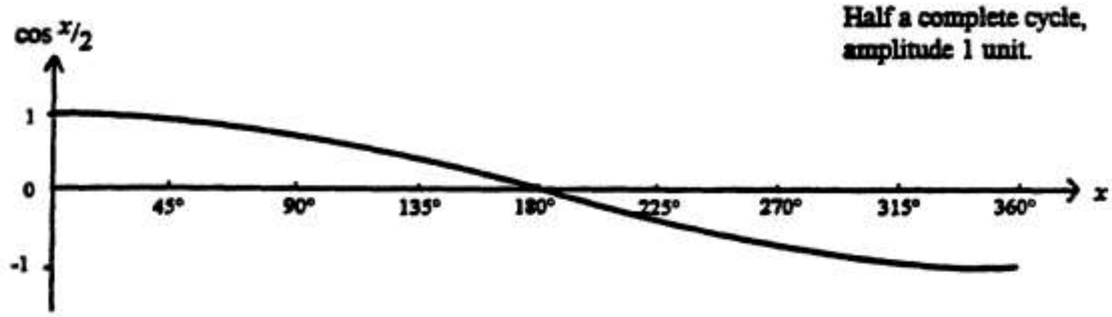
1.



One complete cycle, amplitude 1 unit.



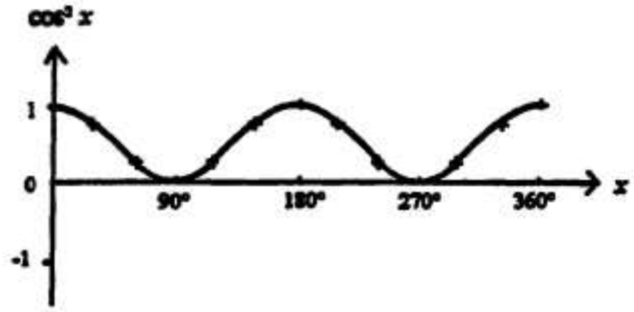
Two complete cycles, amplitude 1 unit.



Half a complete cycle, amplitude 1 unit.

Exercise B

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos^2 x$	1	0.75	0.25	0	0.25	0.75	1	0.75	0.25	0	0.25	0.75	1



Answers

Exercise C

1. $\pi/2$ or 1.57 seconds
2. $2\pi/5$ or 1.26 seconds
3. 8π or 25.13 seconds
4. 16π or 50.27 seconds

Exercise D

1. a) 6amps
   b) $\pi/2$ or 1.57 seconds
   c) 0.64 Hz.
   d) $\pi/4$ radians lagging behind 6 sin4t
2. a) 15amps
   b) 4π or 12.57 seconds
   c) 0.08 Hz.
   d) $\pi/6$ radians leading 15 sin0.5t
3. a) 1.3amps
   b) 1 second
   c) 1 Hz.
   d) $\pi/2$ radians lagging behind 1.3 sin2πt
4. a) 0.8amps
   b) 0.4 seconds
   c) 2.5 Hz.
   d) $\pi/12$ radians leading 0.8 sin5πt
5. a) 25amps
   b) π or 3.14 seconds
   c) 0.32 Hz.
   d) 0.15 radians lagging behind 25sin 2t

Exercise E and F

1. 9.5sin $(\theta + 33.0°)$
2. 4.6sin $(\theta - 27.2°)$
3. 5.0sin $(\theta + 3.0°)$
4. 9.8sin $(\theta - 31.7°)$