Unit 23

Trig identities

Objectives

On completion of this unit you should understand:

1. The trig ratios for angles from $0°$ to $360°$.

2. The ratios of the special angles $30°$, $60°$ and $45°$.

3. The use of the trig ratios $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.

4. The use of the compound angle formulae.

5. The use of the double angle formulae.

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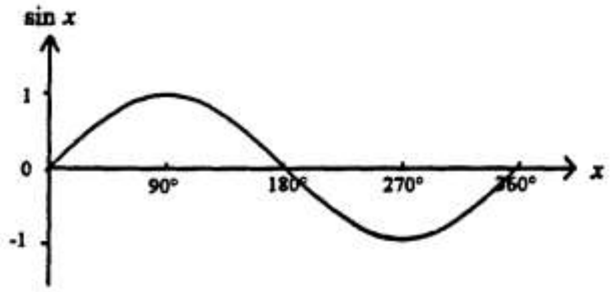
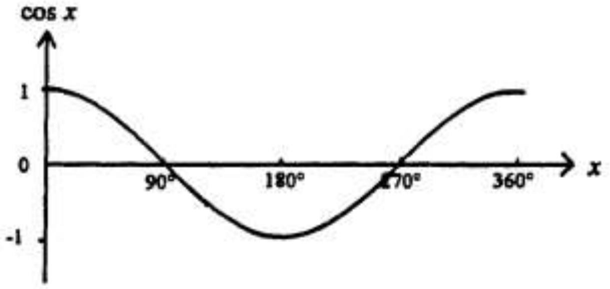
Sin, cos and tan graphs

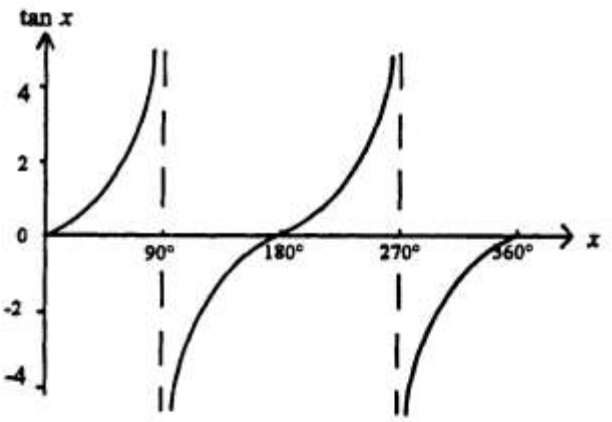
We are now going to compare the three graphs below. Each one is plotted for values of x from 0° to 360°.



You should be able to see that between 0° and 180° the sin graph is positive, but between 180° and 360° it is negative.



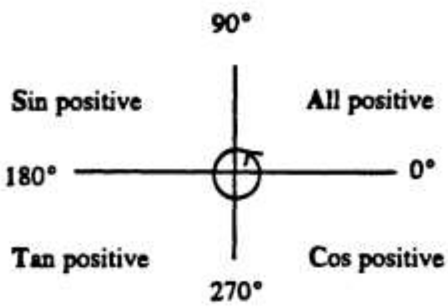
Between 0° and 90° and between 270° and 360° the cos graph is positive. Between 90° and 270° it is negative.

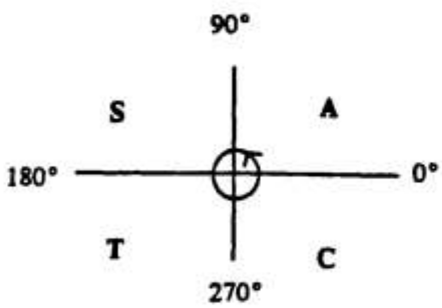


Between 0° and 90° and between 180° to 270° the tan graph is positive. Between 90° to 180° and between 270° and 360° it is negative.

These observations are summerised below in a table and quadrant diagrams.

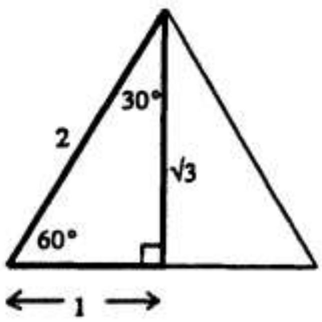
	0° to 90°	90° to 180°	180° to 270°	270° to 360°
sin x	positive	positive	negative	negative
cos x	positive	negative	negative	positive
tan x	positive	negative	positive	negative





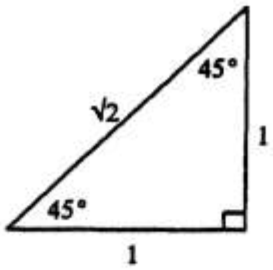
Common trig ratios

We find these from two triangles. The first one is an equilateral triangle of side 2 units. The second is an isosceles triangle.



The equilateral triangle has been divided into two equal parts and the perpendicular height has been found by using the theorem of Pythagoras.
From the triangle we can now find the sin, cos and tan of 60° and 30°.

$\sin 60° = \sqrt{3}/_2$	$\cos 30° = \sqrt{3}/_2$	$\sin 60° = \cos 30°$
$\cos 60° = 1/_2$	$\sin 30° = 1/_2$	$\cos 60° = \sin 30°$
$\tan 60° = \sqrt{3}/_1 = \sqrt{3}$	$\tan 30° = 1/_{\sqrt{3}}$	



We let the two equal sides of a 90° isosceles triangle be 1 unit.
The hypotenuse is found by the theorem of Pythagoras.
We then find the sin, cos and tan of 45° from the triangle.

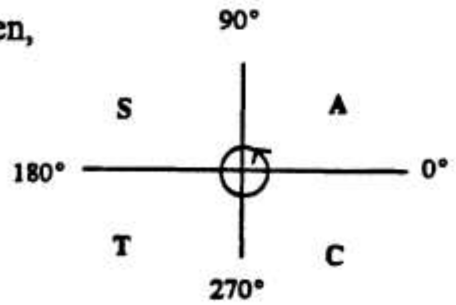
$\sin 45° = 1/_{\sqrt{2}}$ $\cos 45° = 1/_{\sqrt{2}}$ $\tan 45° = 1/_1 = 1$

Study this example.

Example 1

Using the graphs, or the quadrant diagram given, find the ratio for each of the following,

a) sin150°,
b) cos240°,
c) tan315°.



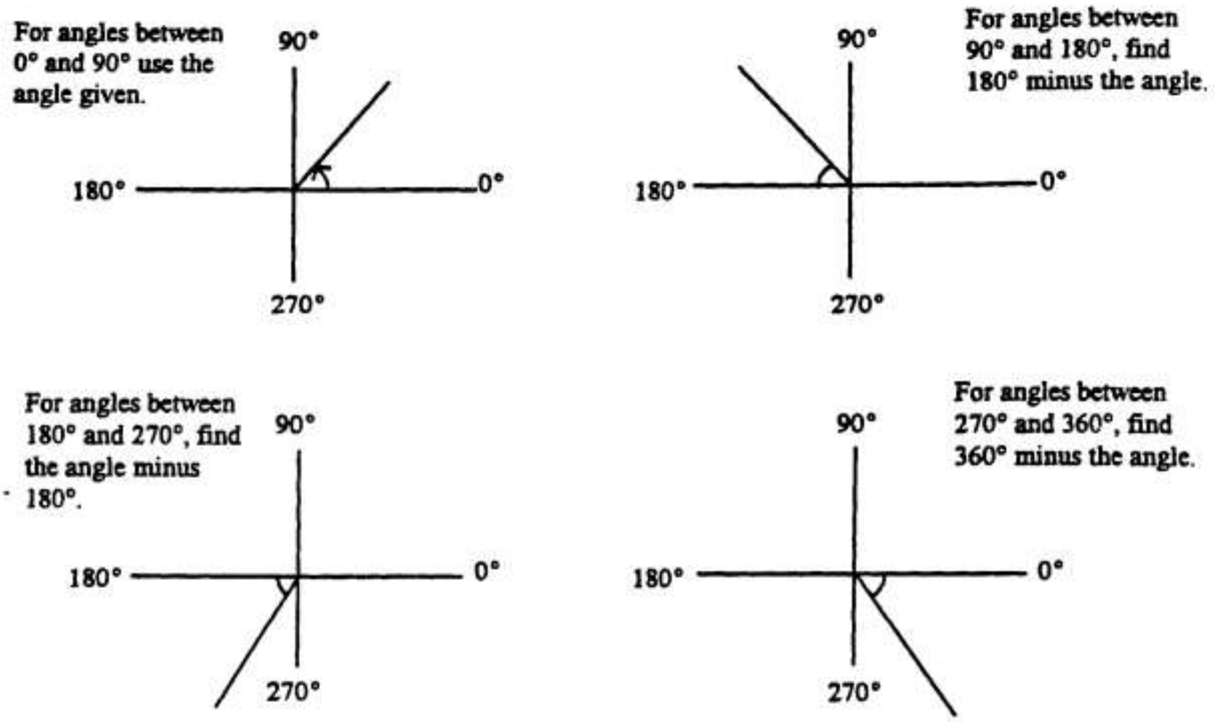
Each time the magnitude of the angle is calculated from the x axis.

a) $\sin 150° = \sin(180° - 150°)$
$= \sin 30° = 1/_2$.

b) This is in a negative quadrant for cos, so, $\cos 240° = -\cos(240° - 180°)$
$\cos 240° = -\cos 60° = -1/_2$.

c) This is a negative quadrant for tan, so, $\tan 315° = -\tan(360° - 315°)$
$\tan 315° = -\tan 45° = -1$.

Remember to find the magnitude of the angle from the x axis. We can summarise this as follows.



Try this exercise.

Exercise A

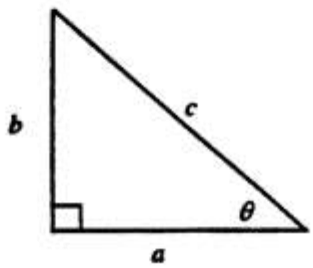
Find the ratio for each of the following angles in fractional form.

1. tan225°
2. sin225°
3. cos330°
4. tan330°
5. sin240°
6. cos135°
7. sin330°
8. cos45°
9. tan150°
10. sin120°

Check your answers with those at the end of the unit.

Trig Identities

Consider the following right angled triangle.



From the triangle,

$\sin\theta = \frac{b}{c}$ $\qquad \cos\theta = \frac{a}{c}$ $\qquad \tan\theta = \frac{b}{a}$

but, $\sin\theta \div \cos\theta = \frac{b}{c} \div \frac{a}{c} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

Using the triangle again,

$\sin\theta = \frac{b}{c}$ $\qquad \cos\theta = \frac{a}{c}$

so, $(\sin\theta)^2 + (\cos\theta)^2 = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{b^2 + a^2}{c^2}$

but, using the thoerem of Pythagoras, $b^2 + a^2 = c^2$

so, $(\sin\theta)^2 + (\cos\theta)^2 = \frac{c^2}{c^2} = 1$

This is usually written as,

$$\sin^2\theta + \cos^2\theta = 1$$

We should also mention at this point that,

$\sec\theta = \frac{1}{\cos\theta}$ $\qquad \text{cosec}\,\theta = \frac{1}{\sin\theta}$ $\qquad \cot\theta = \frac{1}{\tan\theta}$

using these facts we could prove in a similar way that,

$$1 + \tan^2\theta = \sec^2\theta$$

and

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

Study the following examples.

Example 2

If $\sin A = {}^{7}/_{25}$ and A is obtuse, find the value of cos A.

$\sin^2\theta + \cos^2\theta = 1$

$({}^{7}/_{25})^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - ({}^{7}/_{25})^2 = 1 - {}^{49}/_{625} = {}^{625}/_{625} - {}^{49}/_{625} = {}^{576}/_{625}$

$\cos\theta = \pm{}^{\sqrt{576}}/_{\sqrt{625}} = \pm{}^{24}/_{25}$

A is obtuse, it lies between 90° and 180°. This is a negative quadrant for cos, so,

$\cos A = {}^{-24}/_{25}$.

Example 3
If cos A = $^4/_5$ and A is a reflex angle, find the value of sin A.

$\sin^2\theta + \cos^2\theta = 1$
$\sin^2\theta + (^4/_5)^2 = 1$
$\sin^2\theta = 1 - (^4/_5)^2 = 1 - {}^{16}/_{25} = {}^{25}/_{25} - {}^{16}/_{25} = {}^9/_{25}$
$\sin\theta = \pm^{\sqrt{9}}/_{\sqrt{25}} = \pm^3/_5$
A is a reflex angle, it lies between 180° and 360°. These are negative quadrants for sin, so,
sin A = $^{-3}/_5$.

Example 4
If tan A = $^4/_3$ and A lies between 180° and 270°, find the value of cos A.

$1 + \tan^2\theta = \sec^2\theta$
$\sec^2\theta = 1 + (^4/_3)^2$
$\sec^2\theta = 1 + {}^{16}/_9 = {}^9/_9 + {}^{16}/_9 = {}^{25}/_9$
$\sec\theta = \pm^{\sqrt{25}}/_{\sqrt{9}} = \pm^5/_3$
$\cos\theta = \dfrac{1}{\sec\theta} = 1 \div \pm^5/_3 = 1 \times \pm^3/_5 = \pm^3/_5$
A , lies between 180° and 270°. This is a negative quadrant for cos, so,
cos A = $^{-3}/_5$.

Example 5
If sin A = $^3/_5$ and cos A = $^{-4}/_5$, find the value of tan A.

Using the relationship,
$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ $\qquad$ $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{^3/_5}{^{-4}/_5} = {}^3/_5 \div {}^{-4}/_5$

$\tan A = {}^3/_5 \times {}^5/_{-4} = {}^{-3}/_4$.

Try the exercise on the next page.

Exercise B

In this exercise give each answer as a fraction without finding the value of angle A.

1. If sinA = $^4/_5$ and A is an acute angle, find the value of cosA. (An acute angle lies between 0° and 90°).
2. If cosA = $^{-5}/_{13}$ and A lies between 180° and 270° find the value of sinA.
3. If sinA = $^{-3}/_5$ and A lies between 180° and 270°, find the value of cosA.
4. If cosA = $^{24}/_{25}$ and A is a reflex angle, find the value of sinA.
5. If tanA = $^{-24}/_7$ and A is a reflex angle, find the value of cosA.
6. If cosA = $^4/_5$ and A is a reflex angle, find the value of tanA.
7. If cosA = $^{-24}/_{25}$ and sin A = $^7/_{25}$, find the value of tanA.

Check your answers with those at the end of the unit.

Compound angle formulae

These are a set of six formulae, which can be proved and which come in useful for solving problems in science and engineering.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Study these examples.

Example 6

Using a compound angle formula in each case, simplify,

a) $\sin(x - 90°)$,
b) $\cos(270° + x)$.

a) $\sin(A - B) = \sin A\cos B - \cos A\sin B$

so, $\sin(x - 90°) = \sin x \cos 90° - \cos x \sin 90°$

$\sin 90° = 1$ and $\cos 90° = 0$,

so, $\sin(x - 90°) = -\cos x$.

b) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(270° + x) = \cos 270° \cos x - \sin 270° \sin x$

$\cos 270° = 0$ and $\sin 270° = -1$

so, $\cos(270° + x) = \sin x$.

Example 7

Using the sin, cos and tan ratios for 30°, 45° and 60° simplify,

a) $\cos 75°$,
b) $\sin 105°$,
c) $\tan 15°$.

a) $\cos 75° = \cos(30° + 45°)$

Use, $\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(30° + 45°) = \cos 30°\cos 45° - \sin 30°\sin 45°$

$\cos 30° = \sqrt{3}/2 \quad \cos 45° = 1/\sqrt{2} \quad \sin 30° = 1/2 \quad \sin 45° = 1/\sqrt{2}$

$\cos(30° + 45°) = \sqrt{3}/2 \times 1/\sqrt{2} - 1/2 \times 1/\sqrt{2} = \sqrt{3}/2\sqrt{2} - 1/2\sqrt{2}$

$= \frac{1}{2\sqrt{2}}(\sqrt{3} - 1)$ multiply top and bottom by $\sqrt{2}$.

$\cos 75° = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$.

b) $\sin 105° = \sin(60° + 45°)$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(60° + 45°) = \sin 60°\cos 45° + \cos 60°\sin 45°$

$\sin(60° + 45°) = \sqrt{3}/2 \times 1/\sqrt{2} + 1/2 \times 1/\sqrt{2} = \sqrt{3}/2\sqrt{2} + 1/2\sqrt{2}$

$= \frac{1}{2\sqrt{2}}(\sqrt{3} + 1)$ multiply top and bottom by $\sqrt{2}$.

$\sin 105° = \frac{\sqrt{2}}{4}(\sqrt{3} + 1)$.

c) $\tan 15° = \tan(60° - 45°)$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\tan(60° - 45°) = \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$

$\tan 60° = \sqrt{3} \quad \tan 45° = 1$

$$\tan 15° = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

Try this exercise, then check your answers with those at the end of the unit.

Exercise C

Using a compound angle formula in each case, simplify the following.

1. $\sin(360° - x)$
2. $\cos(180° + x)$

Using the sin, cos and tan ratios for 30°, 45° and 60° simplify the following.

3. $\sin 75°$
4. $\sin 15°$
5. $\cos 105°$
6. $\tan 75°$

Study this example.

Example 8

If, $i_1 = 9\cos x$ and $i_2 = 7\sin(x + 45°)$, find the angle x when i_1 is equal to i_2. x is an acute angle.

$$9\cos x = 7\sin(x + 45°) = 7(\sin x \cos 45° + \cos x \sin 45°)$$
$$= 7(1/\sqrt{2} \sin x + 1/\sqrt{2}\cos x)$$
$$9\cos x = 7/\sqrt{2} \sin x + 7/\sqrt{2}\cos x$$
$$7/\sqrt{2} \sin x = \cos x \times (9 - 7/\sqrt{2})$$

Divide through by cos x.

$$\frac{7/\sqrt{2} \sin x}{\cos x} = 9 - 7/\sqrt{2}$$
$$\tan x = \frac{9 - 7/\sqrt{2}}{7/\sqrt{2}}$$
$$\tan x = 0.8182745$$
$$x = 39.3° \text{ to 1 decimal place.}$$

Try this exercise.

Exercise D

Find the angle x in the following questions, giving all answers correct to one decimal place, if x is an acute angle.

1. $4\cos x = 5\sin x$
2. $3\cos x = 5\sin(x - 30°)$
3. $2\sin x = 7\cos(x + 60°)$
4. $5\cos(x - 30°) = 3\sin x$
5. $10\sin(45° + x) = 12\sin x$
6. $6\cos x = 5\cos(x - 60°)$

Check your answers with those at the end of the unit.

The double angle formulae

If we use three of the compound angle formulae, sin(A + B), cos(A + B) and tan(A + B), and let A equal B, then we obtain the following formulae.

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

can be combined with

$$\sin^2 A + \cos^2 A = 1 \text{ so that, } \sin^2 A + \cos^2 A = 1$$

to provide two more useful formulae. These are,

$$\mathbf{\cos 2A = 2\cos^2 A - 1}$$

and $$\mathbf{\cos 2A = 1 - 2\sin^2 A}$$

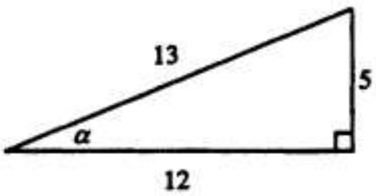
Study this example.

Example 9

α and β are acute angles, $\tan\alpha = {}^5/_{12}$ and $\tan\beta = {}^3/_4$. Without finding the values of α and β, find the values of,

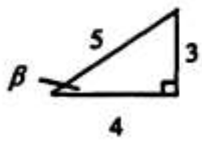
a) $\sin 2\beta$,
b) $\cos 2\alpha$,
c) $\tan 2\alpha$.

We can use right angled triangles and the theorem of Pythagoras to find $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$.



$\sin\alpha = \frac{5}{13}$

$\cos\alpha = \frac{12}{13}$



$\sin\beta = \frac{3}{5}$

$\cos\beta = \frac{4}{5}$

a) $\sin 2\beta = 2\sin\beta\cos\beta$
$= 2 \times {}^3/_5 \times {}^4/_5 = {}^{24}/_{25}.$

b) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= {}^{12}/_{13} \times {}^{12}/_{13} - {}^5/_{13} \times {}^5/_{13} = {}^{119}/_{169}.$

c) $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

$= \dfrac{2 \times {}^5/_{12}}{1 - {}^5/_{12} \times {}^5/_{12}}$

$= \dfrac{{}^5/_6}{1 - {}^{25}/_{144}} = \dfrac{{}^5/_6}{{}^{144}/_{144} - {}^{25}/_{144}}$

$= {}^{120}/_{119}.$

Try the exercise on the next page.

Exercise E

In the following questions α and β are acute angles, $\sin\alpha = {}^{7}/_{25}$ and $\tan\beta = {}^{4}/_{3}$.

Without finding the values of α and β, find the values of,

1. $\sin 2\alpha$
2. $\cos 2\alpha$
3. $\tan 2\alpha$
4. $\sin 2\beta$
5. $\tan 2\beta$
6. $\cos 2\beta$
7. $\sin 2\alpha - \sin 2\beta$
8. $\cos 2\beta + \cos 2\alpha$

- *Check your answers with those at the end of the unit.*

$R\sin(\theta \pm \alpha)$

Consider the first two compound angle formulae,

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

then, $$\mathbf{R\sin(\theta \pm \alpha) = R\sin\theta\cos\alpha \pm R\cos\theta\sin\alpha}$$

Consider the following example.

Example 10

Express $3\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$.

$$3\sin\theta - 4\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

We now compare coefficients.

Coefficients of $\sin\theta$. $\quad$ Coefficients of $\cos\theta$.

$3 = R\cos\alpha$ --------(1) $\quad$ $-4 = -R\sin\alpha$ --------(2)

We can now divide equation (2) by equation (1).

$$\frac{-R\sin\alpha}{R\cos\alpha} = \frac{-4}{3} \quad \text{so,} \quad \tan\alpha = {}^{4}/_{3} \text{ and } \alpha = 53.1°.$$

Using equations (1) and (2) again,

$$\cos\alpha = \frac{3}{R} \text{ so, } \cos^2\alpha = \frac{9}{R^2} \quad \text{and} \quad \sin\alpha = \frac{4}{R} \text{ so, } \sin^2\alpha = \frac{16}{R^2}.$$

We use the identity, $\sin^2\theta + \cos^2\theta = 1$ to find R.

and, $$\sin^2\alpha + \cos^2\alpha = 1$$

$$\frac{16}{R^2} + \frac{9}{R^2} = 1 \quad \text{so,} \quad 16 + 9 = R^2 \quad R = 5$$

so, $$3\sin\theta - 4\cos\theta = 5\sin(\theta - 53.1°).$$

In general if,

$$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$$

then, $R^2 = a^2 + b^2$ and $\tan\alpha = \frac{b}{a}$

Try this last exercise.

Exercise F

In the following questions find R and α correct to two decimal places.

1. Express $3\sin\theta + 5\cos\theta = R\sin(\theta + \alpha)$
2. Express $4\cos\theta + 7\sin\theta = R\sin(\theta + \alpha)$
3. Express $12\sin\theta - 7\cos\theta = R\sin(\theta - \alpha)$
4. Express $-1.2\cos\theta + 2.5\sin\theta = R\sin(\theta - \alpha)$
5. Express $-2\cos\theta + 6\sin\theta = R\sin(\theta \pm \alpha)$

Check your answers with those at the end of the unit.

Answers

Exercise A

1. 1
2. $^1/_{\sqrt{2}}$
3. $^{\sqrt{3}}/_2$
4. $^{-1}/_{\sqrt{3}}$
5. $^{-\sqrt{3}}/_2$
6. $^{-1}/_{\sqrt{2}}$
7. $^{-1}/_2$
8. $^1/_{\sqrt{2}}$
9. $^{-1}/_{\sqrt{3}}$
10. $^{\sqrt{3}}/_2$

Exercise B

1. $^3/_5$
2. $^{-12}/_{13}$
3. $^{-4}/_5$
4. $^{-7}/_{25}$
5. $^7/_{25}$
6. $^{-3}/_4$
7. $^{-7}/_{24}$

Exercise C

1. $-\sin x$
2. $-\cos x$
3. $\dfrac{\sqrt{2}(1+\sqrt{3})}{4}$
4. $\dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$
5. $\dfrac{\sqrt{2}(1-\sqrt{3})}{4}$
6. $\dfrac{1+\sqrt{3}}{\sqrt{3}-1}$

Exercise D

1. 38.7°
2. 51.8°
3. 23.5°
4. 83.4°
5. 55.1°
6. 38.9°

Exercise E

1. $^{336}/_{625}$
2. $^{527}/_{625}$
3. $^{336}/_{527}$
4. $^{24}/_{25}$
5. $^{-24}/_7$
6. $^{-7}/_{25}$
7. $^{-264}/_{625}$
8. $^{352}/_{625}$

Exercise F

1. $5.83\sin(\theta + 59.04°)$
2. $8.06\sin(\theta + 29.74°)$
3. $13.89\sin(\theta - 30.26°)$
4. $2.77\sin(\theta - 25.64°)$
5. $6.32\sin(\theta - 18.43°)$