Unit 22

Radians

Objectives

On completion of this unit you should understand:

1. Angular measure in degrees and radians.

2. Conversion of degree measure to radians and vice versa.

3. Use the relationship $s = r\theta$ for the length of an arc of a circle.

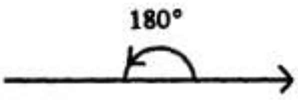
4. Use the relationship $A = \frac{1}{2}(r^2\theta)$ for the area of a sector of a circle.

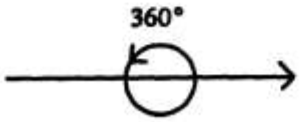
5. Use the relationship $A = \frac{1}{2}r^2(\theta - \sin\theta)$ for the area of a segment of a circle.

Cook, M. and Rimmer, H. (1993),
Stafford College, Earl Street, Stafford, ST16 2QR

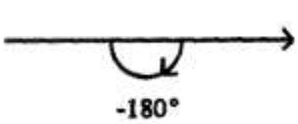
Angular measurement

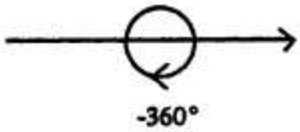
We need to consider an angular measurement as a rotation. Half a revolution in an anticlockwise direction is 180°. A quarter of a revolution is 90°. A complete revolution is 360°. We can continue in this way. For example two complete anticlockwise revolutions would then be 360° x 2 = 720°.





In a similar way we say that a clockwise rotation of one complete revolution is -360°. Half a clockwise revolution is -180° and so on.





Study this example.

Example 1

Write the number of degrees represented by,

a) $1\frac{1}{2}$ revolutions in an anticlockwise direction,

b) 2.5 revolutions in a clockwise direction.

a) $1\frac{1}{2}$ revolutions in an anticlockwise direction = $1\frac{1}{2}$ x 360° = 540°.

b) 2.5 revolutions in a clockwise direction = 2.5 x -360° = -900°.

Try this exercise.

Exercise A

Write the number of degrees represented by the following rotations.

1. 3 revolutions anticlockwise.
2. 1.5 revolutions clockwise.
3. 2.5 revolutions anticlockwise.
4. 5 revolutions anticlockwise.
5. $1\frac{1}{4}$ revolutions clockwise.
6. $1\frac{3}{4}$ revolutions clockwise.
7. 0.75 revolutions anticlockwise.
8. 4 revolutions clockwise.
9. $2\frac{1}{4}$ revolutions clockwise.
10. $3\frac{3}{4}$ revolutions anticlockwise.

Check your answers with those at the end of the unit.

Each degree can be divided into 60 minutes. We can write this as 60'.
Study this example.

Example 2
Convert 45.5° to degrees and minutes.

If there are 60' in each degree, then 0.5° = 30'.
45.5° = 45° 30'.
This can be done using your calculator.
There is a button which looks like this **°′″**.

Calculator: 45.5 INV °′″ 45° 30° 0
We write this as 45° 30'.

The final figure in the calculator display is seconds. There are 60 seconds in each minute. We could write 45° 30' 0".

Example 3
Using your calculator convert 37° 27' to degrees.

Calculator: 37 °′″ 27 °′″ 37.45
37° 27' = 37.45°.

Try this exercise using your calculator.

Exercise B
Convert the following to degrees and minutes.
1. 138.7°
2. 168.4°
3. 39.45°
4. 18.2°
5. -312.99°

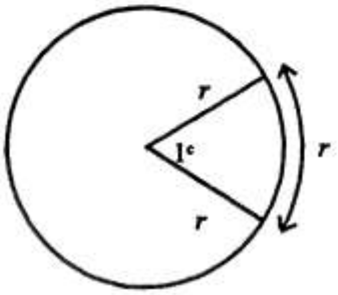
Convert the following to degrees. Give answers correct to 2 decimal places.

6. 19° 55'
7. 25° 52'
8. 33° 15'
9. 239° 35'
10. 29° 47'

Check your answers with those at the end of the unit.

The relationship between degrees and radians

We have already measured angles in degrees. There is a second way of measuring an angle. This method involves the use of the radian.



A radian is the angle subtended at the centre of a circle by an arc of length r, where r is the radius of the circle.

We write 1^c.
This means 1 radian.

The circumference of a circle is $2\pi r$.
The number of radians in one complete revolution is therefore

$$\frac{2\pi r}{r} = 2\pi \text{ radians.}$$

This means that

$$2\pi^c \equiv 360°$$

and

$$\pi^c \equiv 180°.$$

Study these examples.

Example 4
Convert 30° to radians.

$$180° \equiv \pi^c$$

so, $$1° \equiv \frac{\pi}{180} \text{ radians}$$

$$30° \equiv 30 \times \frac{\pi}{180} = \frac{30}{180} \times \pi$$

$$= \frac{1}{6} \times \pi$$

$$= \frac{\pi}{6} \text{ radians.}$$

We call this 'π by 6'.

Example 5
Convert -270° to radians.

$$180^\circ \equiv \pi^c$$

so, $$1^\circ \equiv \frac{\pi}{180} \text{ radians}$$

$$-270^\circ \equiv -270 \times \frac{\pi}{180} = \frac{-270}{180} \times \pi$$

$$= \frac{-3}{2} \times \pi$$

$$= \frac{-3\pi}{2} \text{ radians.}$$

$^3/_2$ is left as a 'top heavy' fraction. We call $^{-3\pi}/_2$, 'minus 3π by 2'.

Example 6
Convert 43° to radians.

$$180^\circ \equiv \pi^c$$

so, $$1^\circ \equiv \frac{\pi}{180} \text{ radians}$$

$$43^\circ \equiv 43 \times \frac{\pi}{180} = \frac{43}{180} \times \pi$$

43 ÷ 180 cannot be cancelled into a simple fraction, so we use the calculator and find that to three significant figures, $43^\circ = 0.750^c$.

Try this exercise.

Exercise C
Convert each of the following to radians. Express answers as a fraction of π.

1. 60°	4. -10°	7. -150°
2. -45°	5. 90°	8. 210°
3. 20°	6. 720°	9. -330°

Using your calculator express each of the following in radians, correct to 2 decimal places.

10. 339°	13. 34°	16. 420°
11. -221°	14. -56°	17. -331°
12. -23°	15. 99°	18. 17°

Check your answers with those at the end of the unit.

We shall now convert radians to degrees in a similar way.
Study these examples.

Example 7
Convert $\pi/2^c$ to degrees.

$$\pi^c \equiv 180^\circ$$
$$1^c \equiv \frac{180}{\pi} \text{ degrees}$$
$$\frac{\pi}{2}^c = \frac{180}{\pi} \times \frac{\pi}{2} = 90^\circ.$$

Example 8
Convert $-3\pi/2^c$ to degrees.

$$\pi^c \equiv 180^\circ$$
$$1^c \equiv \frac{180}{\pi} \text{ degrees}$$
$$-\frac{3\pi}{2}^c = \frac{180}{\pi} \times -\frac{3\pi}{2} = -270^\circ.$$

Example 9
Convert 3.185^c to degrees. Answer to one decimal place.

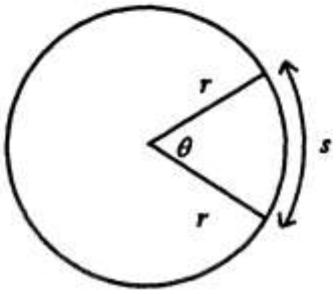
$$\pi^c \equiv 180^\circ$$
$$1^c \equiv \frac{180}{\pi} \text{ degrees}$$
$$3.185^c = \frac{180}{\pi} \times 3.185 = 182.5^\circ.$$

Try this exercise.

Exercise D
Convert the following radians to degrees. Answer to one decimal place where appropriate.

1. $\pi/4^c$
2. $\pi/6^c$
3. $-2\pi/3^c$
4. 0.567^c
5. -4.523^c
6. $5\pi/4^c$
7. $-7\pi/4^c$
8. $2\pi^c$
9. -3.599^c
10. 2.709^c

Length of the arc of a circle



The length of the arc is represented by s.
The arc subtends an angle θ in radians at the centre of the circle.
The circumference of a circle is $2\pi r$.
One complete revolution is $2\pi^c$.
So,

$$\frac{\theta}{2\pi} = \frac{s}{2\pi r}$$

and,

$$\boldsymbol{s = r\theta}$$

where θ is measured in radians.

Study these examples.

Example 10
Find the length of the arc of a circle, radius 4cm. if the arc subtends an angle of 0.73 radians at the centre of the circle.

Using $s = r\theta$
$s = 4 \times 0.73$
$s = 2.92$cm.

Example 11
An arc, length 2.7cm. subtends an angle θ at the centre of a circle, radius 5cm.
Find the value of θ.

Using $s = r\theta$
$2.7 = 5 \times \theta$
$\theta = \frac{2.70}{5} = 0.54^c$.

Example 12
An arc, length 7.8cm. subtends an angle $\pi/6^c$ at the centre of a circle, radius $r$cm. Find the value of r to one decimal place.

Using $s = r\theta$
$7.8 = r \times \pi/6$
$r = \frac{7.8}{\pi/6} = 14.9$cm.

Now try the following exercise.

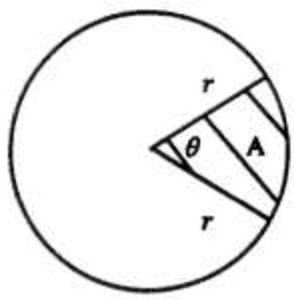
Exercise E

Give answers correct to 2 decimal places.

1. Find the length of the arc of a circle of radius 6cm. if the arc subtends an angle of 0.33 radians at the centre of the circle.
2. Find the length of the arc of a circle of radius 10cm. if the arc subtends an angle of $\pi/8$ radians at the centre of the circle.
3. Find the length of the arc of a circle of radius 10πcm. if the arc subtends an angle of $\pi/3$ radians at the centre of the circle.
4. An arc, length 8.7cm. subtends an angle θ at the centre of a circle of radius 15cm. Find the value of θ.
5. An arc, length 2.4cm. subtends an angle θ at the centre of a circle of radius 9cm. Find the value of θ.
6. An arc, length 12.8cm. subtends an angle $\pi/3^c$ at the centre of a circle of radius $r$cm. Find the value of r.
7. An arc, length 10.8cm. subtends an angle $2\pi/5^c$ at the centre of a circle of radius $r$cm. Find the value of r.

Check your answers with those at the end of the booklet.

Area of a sector of a circle



The area of the sector, angle θ is represented by A.

The area of a circle is πr^2.

One complete revolution is $2\pi^c$.

So,

$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

and so,

$$A = \frac{1}{2} r^2 \theta$$

where θ is measured in radians.

Study the examples on the next page.

Example 13
The radius of a circle is 4m. Find the length of the arc and area of the sector which has an angle at the centre of 57°. Give answers correct to two decimal places.

Note that since the angle is not given in radians we must first convert it, since the formulae require the angle to be in radians.

$$57° = 0.9948376^c$$
$$s = r\theta$$
$$= 4 \times 0.9948376$$
$$= 3.98\text{m}.$$

$$A = \frac{1}{2}r^2\theta$$
$$= \tfrac{1}{2} \times 4^2 \times 0.9948376$$
$$= 7.96\text{m}^2.$$

Example 14
A sector of area 45mm². subtends an angle of 60° at the centre of the circle. Find the radius of the circle to two decimal places.

We change degrees to radians first.

$$60° = \pi/3 \text{ radians}$$
$$A = \frac{1}{2}r^2\theta$$
$$45 = \tfrac{1}{2} \times r^2 \times \pi/3$$
$$r^2 = \frac{45}{\tfrac{1}{2} \times \pi/3}$$
$$r = \sqrt{\frac{45}{\tfrac{1}{2} \times \pi/3}}$$

Calculator: 45 ÷ (1 a/b 2 x (π ÷ 3)) = √ 9.2705808

$$r = 9.27\text{mm}.$$

Example 15

An arc of a circle subtends an angle θ at its centre. The length of the arc is 5πcm. and the area of the sector subtending angle θ is $45\pi cm^2$. Find the radius of the circle, r, and the value of θ.

The length of the arc is given by,

$$s = r\theta$$

so, $5\pi = r\theta$ ---------------------------(1)

The area of the sector is given by,

$$A = \frac{1}{2}r^2\theta$$

so, $45\pi = \frac{1}{2}r^2\theta$ ---------------------------(2)

We can substitute for $r\theta$ from equation (1) into equation (2).

$$45\pi = \frac{1}{2} \times r \times 5\pi$$

Multiply both sides by 2,

$$90\pi = r \times 5\pi$$

so, $r = \frac{90\pi}{5\pi}$

$r = 18$cm.

We can now find θ.

$$s = r\theta$$

$$5\pi = 18 \times \theta$$

$\theta = \frac{5\pi}{18} = 0.873^c$ to three significant figures.

Try this exercise.

Exercise F

In each case, A represents the area of a sector subtending an angle θ. The length of the arc is s and the radius is r. Find the missing items in each case. Answer to three significant figures, where appropriate.

	A	θ	s	r
1.	?	0.621^c	?	5cm.
2.	?	60°	?	1m.
3.	$20\pi cm^2$.	$\pi/3^c$	?	?
4.	$27m^2$.	?	5m.	?
5.	?	$\pi/6^c$	$5\pi/6$cm.	?

Check your answers with those at the end of the unit.

Area of a triangle

You should already know one formula for calculating the area of a triangle.

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

There is another formula which can be useful if two adjacent sides and the angle between them is known.
Study this diagram.

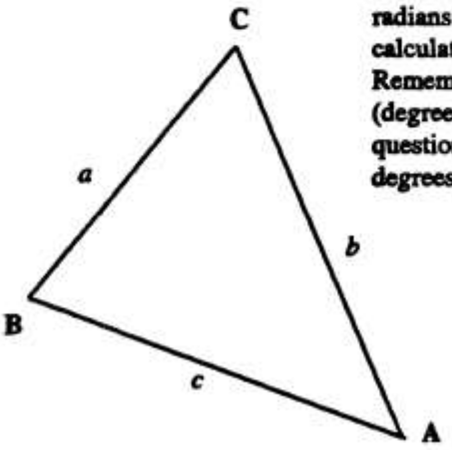
Area of triangle $= \frac{1}{2}ab\sin C$

or,

Area of triangle $= \frac{1}{2}bc\sin A$

or,

Area of triangle $= \frac{1}{2}ac\sin B$



The angle can be given in degrees or radians. If the angle is given in radians you must switch your calculator to RAD mode, MODE 5. Remember to switch it back to DEG (degrees) MODE 4 if the next question has the angle given in degrees.

Example 16

Find the area of the triangle shown in the diagram above, if, a = 6cm., b = 8cm. and angle C = 1.4^c.

$$\textbf{Area} = \frac{1}{2}ab\sin C$$

The area of the triangle = $\frac{1}{2} \times 6 \times 8 \times \sin 1.4^c$

The angle is given in radians, so you need to switch your calculator to radian mode. This is usually **MODE 5**.

Calculator: 1 ÷ 2 x 6 x 8 x 1.4 sin = 23.650794

The area of the triangle is 23.7cm^2. to one decimal place.

Try this exercise.

Exercise G

Find the area of each of the following triangles. Give your answers correct to the nearest whole number.

	a	b	c	A	B	C
1.	5mm.		4mm.		35°	
2.		6.2m.	5.3m.	1.12^c		
3.	12.3cm.	9.8cm.				38°
4.		1.5ft.	2.6ft.	71°		
5.	3.9cm.		2.8cm.		0.471^c	

Check your answers with those at the end of the unit.

Area of a segment

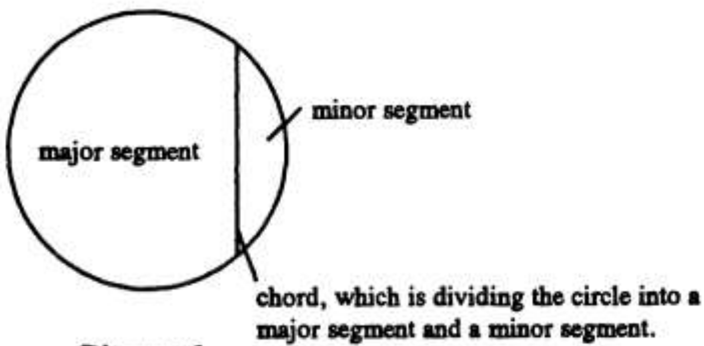


Diagram 1

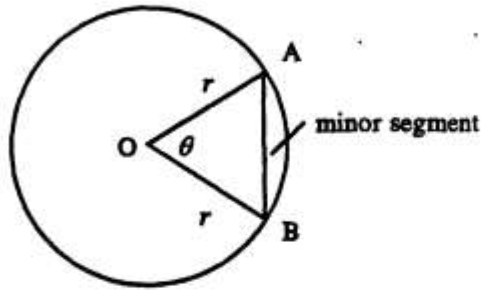


Diagram 2

Diagram 1 shows a circle divided into two segments. It can be seen from diagram 2, that, the area of the minor segment is the area of the sector, AOB minus the area of the triangle AOB. If we use the formula for the area of a triangle given on the previous page, $a = r$ and $b = r$, so we have,

$$\text{Area of segment} = \frac{1}{2}r^2\theta - \frac{1}{2} \times r \times r \times \sin\theta = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$\textbf{Area of segment} = \frac{1}{2}r^2(\theta - \sin\theta)$$

θ must be in radians!

Study this example.

Example 17
Using diagram 2 at the top of the page, find the area of the minor segment if θ is equal to $\pi/4$ radians and r is equal to 4cm.

$$\textbf{Area of segment} = \frac{1}{2}r^2(\theta - \sin\theta)$$

$$\text{Area of segment} = \frac{1}{2} \times 4^2 \times (\pi/4 - \sin\pi/4)$$

Switch your calculator to RAD mode.
Calculator: 0.5 x 4 x^2 x (π ÷ 4 - (π ÷ 4) sin) = 0.626331
The area of the segment is 0.626cm^2. to three significant figures.

Try this short exercise.

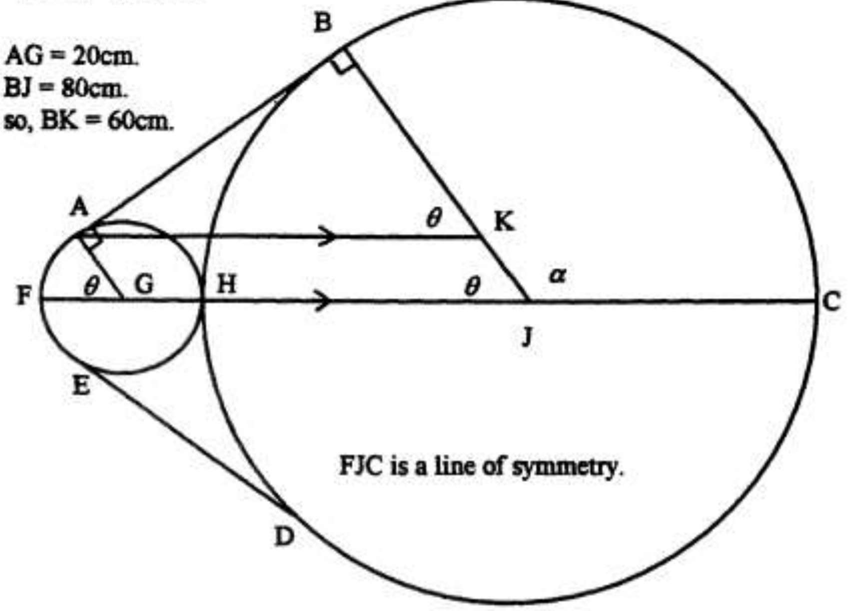
Exercise H
Using diagram 2, find the area of the following segments, to 3 sig. figs.

1. $r = 5$cm. $\theta = 1.2^c$
2. $r = 10$m. $\theta = 0.92^c$
3. $r = 8$cm. $\theta = 1.93^c$
4. $r = 9$m. $\theta = 30°$

Check your answers with those at the end of the unit.

Study this last example.

Example 18

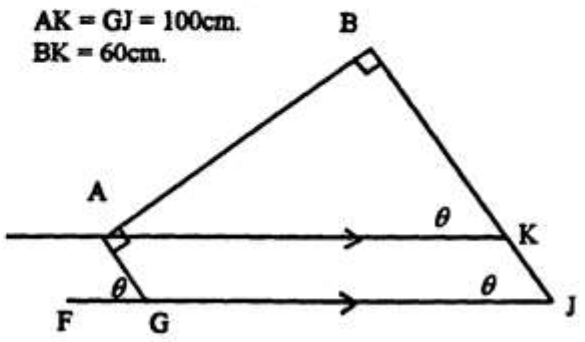


The diagram shows two circular drive wheels centres G and J, radii 20cm. and 80cm., which touch at a point H. A fan belt ABCDEF is stretched round the wheels as shown. Note that the fan belt is a tangent to each of the circles, so is at 90°, ($^{\pi}/_{2}{}^{c}$), to the radii.

Find,

a) AB,
b) the length of the fan belt in contact with the small wheel to 1 d.p.,
c) the length of the fan belt in contact with the large wheel to 1 d.p.,
d) the total length of the belt to three significant figures.

a) Using triangle ABK,
by Pythagoras, AB = 80cm.



b) $\cos\theta = \dfrac{60}{100}$

so θ = 0.9272952 radians.
Store this in the calculator.

Arc AF = $r\theta$
= 20 x 0.9272952
= 18.545904cm.
Arc EFA = 18.545904 x 2
= 37.091809cm.

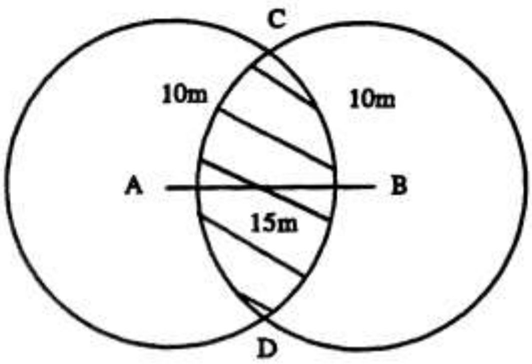
c) $\alpha = 2\pi - \theta$ = 5.3558901
Arc BC = $r \times \alpha$ = 80 x 5.3558901 = 428.47121cm.
Arc DCB = 428.47121 x 2 = 856.94241cm.

d) Total belt = AB + arc AFE + ED + arc BCD = 1054.0342cm.
= 1050cm. to three significant figures.

Try this final exercise.

Exercise I

1. A model railway track is laid in the formation of two circles, centres A and B as shown. The distance between the centre of the circles is 15m. Each circle is radius 10m. Shrubbery is to be planted in the shaded area.



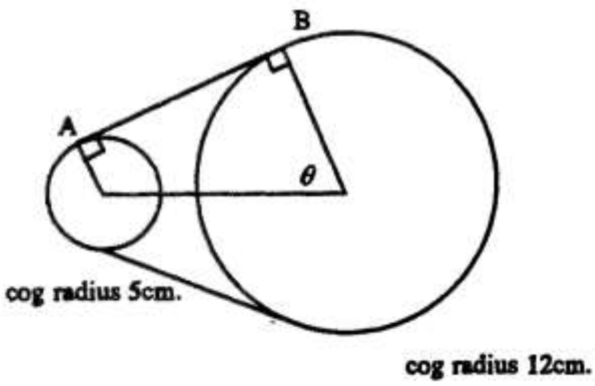
Radii,
AC = CB = 10m.

Distance between centres,
AB = 15m.

Find, to 3 significant figures,

a) the length of CD,
b) angle CAD in radians,
c) the area of the sector CAD,
d) the area of the triangle CAD,
e) the area of the minor segment of the circle centre A,
f) the area to be planted with shrubs.

2. A child's toy truck is driven by a chain round two cogs as shown. The cogs have radii 5cm. and 12cm. The centres of the cogs are 20cm. apart.



Calculate, to 3 significant figures,

a) AB,
b) angle θ, in radians,
c) the length of the chain in contact with the smaller cog.

Check your answers with those at the end of the unit.

Answers

Exercise A

1. 1080°
2. -540°
3. 900°
4. 1800°
5. -450°
6. -630°
7. 270°
8. -1440°
9. -810°
10. 1350°

Exercise B

1. 138° 42'
2. 168° 24'
3. 39° 27'
4. 18° 12'
5. -312° 59'
6. 19.92°
7. 25.87°
8. 33.25°
9. 239.58°
10. 29.78°

Exercise C

1. $\pi/3^c$
2. $-\pi/4^c$
3. $\pi/9^c$
4. $-\pi/18^c$
5. $\pi/2^c$
6. $4\pi^c$
7. $-5\pi/6^c$
8. $7\pi/6^c$
9. $-11\pi/6^c$
10. 5.92^c
11. -3.86^c
12. -0.40^c
13. 0.59^c
14. -0.98^c
15. 1.73^c
16. 7.33^c
17. -5.78^c
18. 0.30^c

Exercise D

1. 45°
2. 30°
3. -120°
4. 32.5°
5. -259.1°
6. 225°
7. -315°
8. 360°
9. -206.2°
10. 155.2°

Exercise E

1. 1.98cm.
2. 3.93cm.
3. 32.90cm.
4. 0.58^c
5. 0.27^c
6. 12.22cm.
7. 8.59cm.

Exercise F

1. $A = 7.76cm^2$. $s = 3.11cm$.
2. $A = 0.524m^2$. $s = 1.05m$. or $(\pi/3)m$.
3. $s = 11.5cm$. $r = 11.0cm$.
4. $r = 10.8m$. $\theta = 0.463^c$
5. $r = 5cm$. $A = 6.54cm^2$.

Exercise G

1. $6mm^2$.
2. $15m^2$.
3. $37cm^2$.
4. $2ft^2$.
5. $2cm^2$.

Exercise H

1. $3.35cm^2$.
2. $6.22m^2$.
3. $31.8cm^2$.
4. $0.956m^2$.

Exercise I

1. a) 13.2m. b) 1.45^c
   c) $72.3m^2$. d) $49.6m^2$.
   e) $22.7m^2$. f) $45.3m^2$.
2. a) 18.7cm. b) 1.21^c
   c) 12.1cm.