Unit 21

Pythagoras, angles of elevation and depression, three dimensional geometry

Objectives

On completion of this unit you should understand:

1. The theorem of Pythagoras.

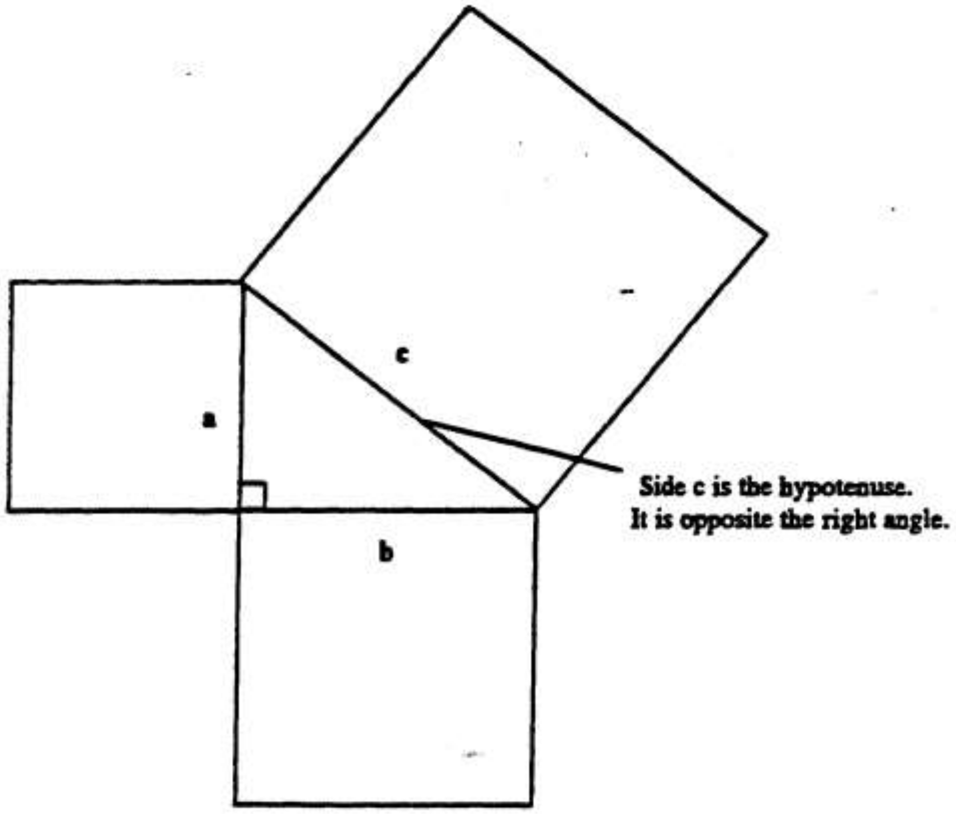
2. Angles of elevation and depression.

3. Three dimensional problems involving Pythagoras and the sine and cosine rules.

Cook, M. and Rimmer, H. (1993),
Stafford College, Earl Street, Stafford, ST16 2QR

Theorem of Pythagoras

Consider the right angled triangle below.



Theorem of Pythagoras

In a right angled triangle, the area of the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This means, that the area of the square drawn on side c, in the diagram, is equal to the sum of the areas of the squares, drawn on sides a and b.
If we write this as a formula,
the area of the square, side **a**, will be, **a x a**,
the area of the square, side **b**, will be, **b x b**,
the area of the square, side **c**, will be, **c x c**,
so we have,

$$a \times a + b \times b = c \times c$$

We can write this simply as,

$$a^2 + b^2 = c^2$$

This is the theorem of Pythagoras and applies to right angled triangles only.

Now consider these examples.

Example 1
Using the theorem of Pythagoras,

$$a^2 + b^2 = c^2$$

if, a right angled triangle has side a = 4cm. and side b = 3cm., calculate the length of side c.

$$a^2 + b^2 = c^2$$

Substituting for a and b,

$$4^2 + 3^2 = c^2$$
$$c^2 = 16 + 9$$
$$c^2 = 25$$

If, c^2 or $c \times c = 25$
we need to find the value of c.
We do this by taking the square root of both sides of the equation. The square root of c^2 is c. We need to find the square root of 25. This can be found using the square root button on your calculator.

Calculator: 25 √ 5

The length of side c is 5cm.

This is a common triangle and is referred to as the 3:4:5 right angled triangle.

Example 2
Use the theorem of Pythagoras to calculate the hypotenuse of a right angled triangle whose other two sides measure 3ft. and 7ft. Answer to one decimal place.

The hypotenuse is always the longest side. Using the formula,

$$a^2 + b^2 = c^2$$
$$3^2 + 7^2 = c^2$$

There is a button on your calculator which will square terms for you. It looks like this x^2.
Using your calculator,
Calculator: 3 x^2 + 7 x^2 = 58
This means that $c^2 = 58$. Continue by pressing the square root button.
Calculator: 58 √ 7.6157731
To one decimal place, the hypotenuse is 7.6ft.

Try this exercise.

Exercise A

Use the theorem of Pythagoras,

$$a^2 + b^2 = c^2$$

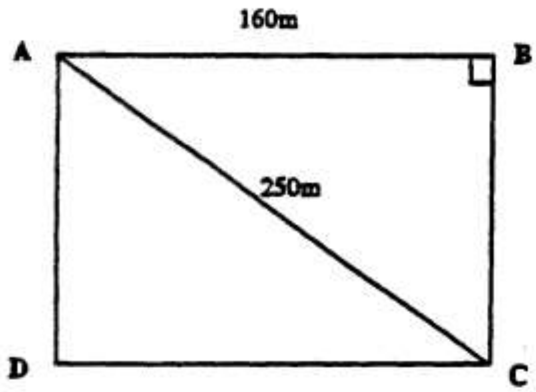
to calculate the hypotenuse of each of the following right angled triangles. Give your answers correct to one decimal place, where appropriate.

	a	b
1.	6m.	5m.
2.	5ft.	12ft.
3.	8in.	15in.
4.	7m.	24m.
5.	11cm.	14cm.
6.	2km.	2km.
7.	6mm.	8mm.
8.	1.2m.	0.5m.
9.	4.7ft.	1.3ft.
10.	3.5cm.	5.2cm.

Check your answers with those at the end of the unit.
Now study this example.

Example 3

The diagram below shows a rectangular field ABCD. The diagonal AC measures 250m. AB is 160m. Calculate the length of BC, to three significant figures.



Pythagoras

$$AC^2 = AB^2 + BC^2$$

$$250^2 = 160^2 + BC^2$$

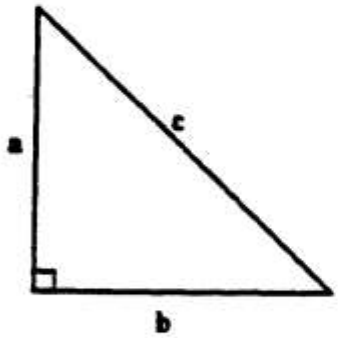
Take 160^2 from both sides.

$$BC^2 = 250^2 - 160^2$$
$$BC^2 = 36900$$
$$BC = 192.09373$$

BC is 192m. to three significant figures.

Try this simple exercise.

Exercise B



Given the triangle above, calculate the size of the missing side in each case. Answer to one decimal place, where appropriate.

	a	b	c
1.	3cm.	?	5cm.
2.	8cm.	?	10cm.
3.	?	12cm.	13cm.
4.	7m.	?	25m.
5.	24m.	?	26m.
6.	8mm.	?	15mm.
7.	?	10m.	14m.
8.	4cm.	?	7cm.
9.	?	12cm.	16cm.
10.	6m.	?	15m.

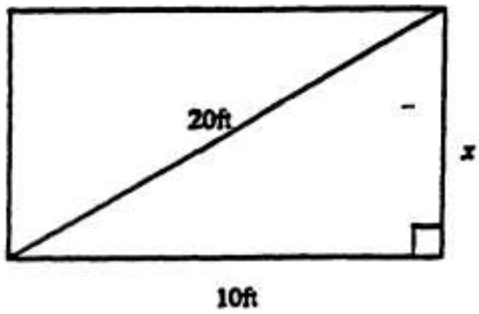
Check your answers with those at the end of the unit.

Now try this exercise. Remember, the hypotenuse is the longest side, and it is always the side opposite the right angle.

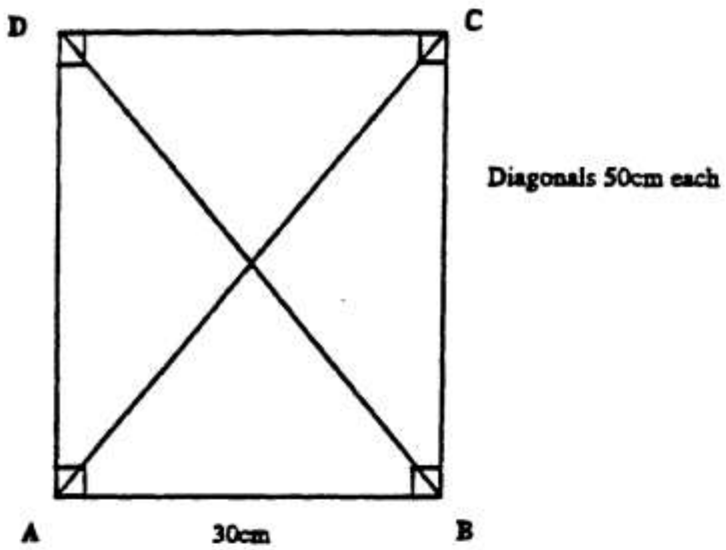
Exercise C

The following diagrams are not drawn to scale. Give answers correct to one decimal place where necessary. Use the theorem of Pythagoras in each case.

1. Find the side marked x.



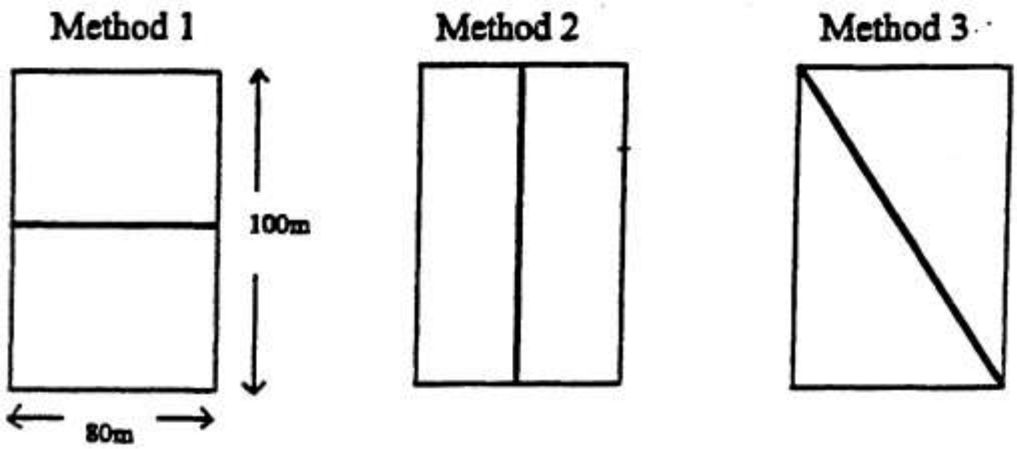
2. The diagram below shows a section of a trellis, made up of six struts. Calculate,
   a) the length of BC,
   b) the total length of wood required to build this section of trellis.



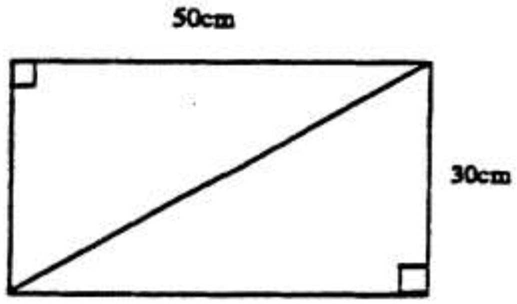
Exercise C is continued on the next page.

Exercise C (*Continued*)

3. A builder decides to divide a field into two equal building plots, by erecting a fence. The original field measures 100m x 80m. Calculate, the length of fence required by each of the three methods.

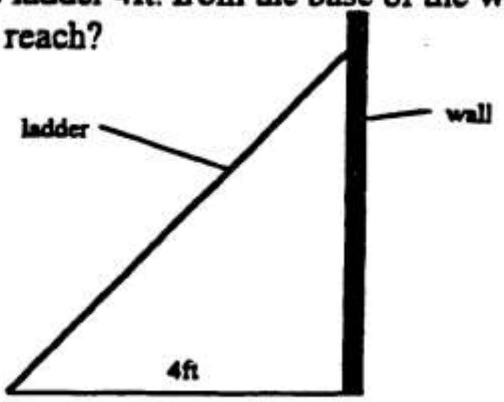


4. The front of a small kiln measures 50cm. x 30cm.



Calculate the length of a diagonal line drawn as shown.

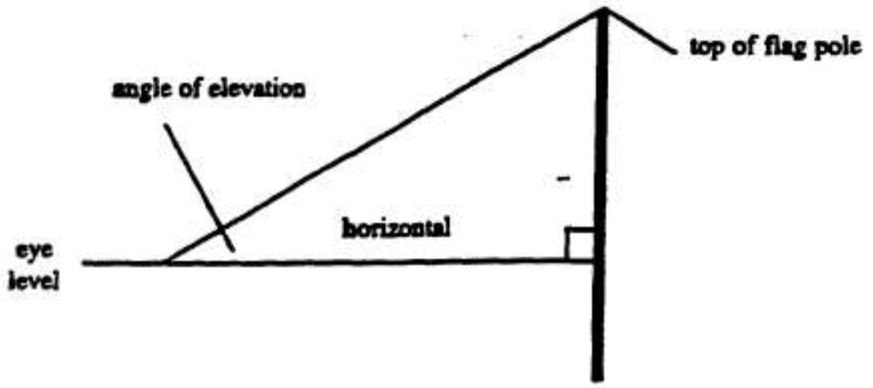
5. If a ladder 12ft. long is rested against a vertical wall as shown, with the foot of the ladder 4ft. from the base of the wall, how far up the wall will the ladder reach?



Check your answers with those at the end of the unit.

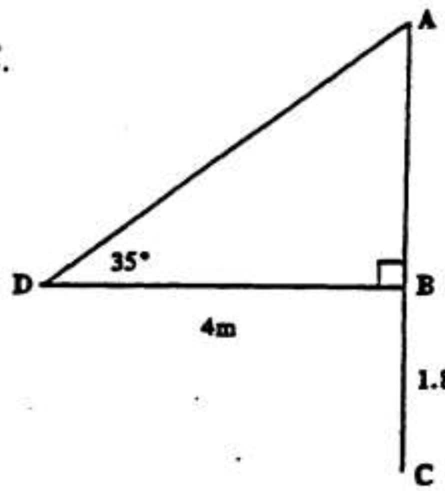
Angles of elevation

If standing on the ground, you look up to the top of a flag pole, the angle of elevation is the angle formed between the horizontal and the line of sight.



Study this example.

Example 4
Using the diagram above, the angle of elevation of the top of the flag pole from a man's eye level is 35°. The man's eye is 4m. horizontally from the flag pole. If the height of the man's eye level is 1.8m. above the ground, calculate the height of the flag pole, to three significant figures.



The height of the flag pole is AB + BC.
We need to find AB.
In triangle ABD,
AB is opposite the 35° angle.
BD is adjacent to the 35° angle.
Use O and A. Use TOA.

$$\tan 35° = \frac{AB}{4}$$

Multiply both sides by 4.

$$AB = 4 \times \tan 35°$$

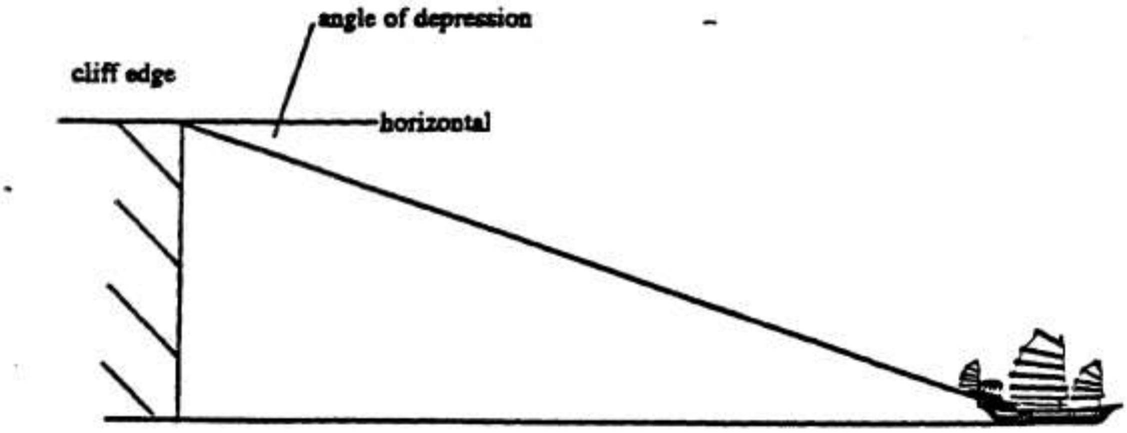
Using the calculator,

$$AB = 2.8008302$$

The height of the flag pole is 1.8 + 2.8008302 = 4.6008302
This is 4.60m. to three significant figures.

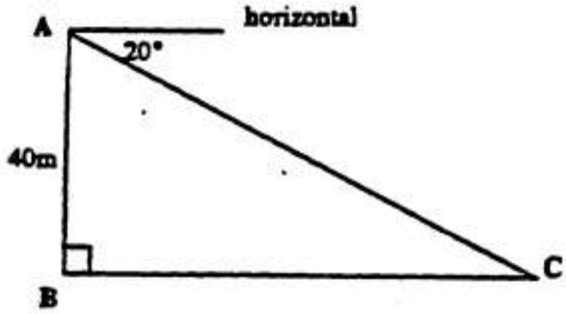
Angles of depression

If you are standing on a cliff and look out to sea at a boat, the angle of depression is the angle formed between the horizonal and the line of sight.



Example 5

In the diagram above, the height of the cliff is 40m. The angle of depression is 20°. How far is the boat from the base of the cliff? Answer to the nearest whole number.



If the angle of depression is 20°,
then angle BAC = 90° - 20° = 70°

In triangle ABC,
AB is adjacent to the 70° angle,
We need to find the side BC.
This is opposite the 70° angle.
Use O and A. Use TOA.

$$\tan 70^\circ = \frac{BC}{40}$$

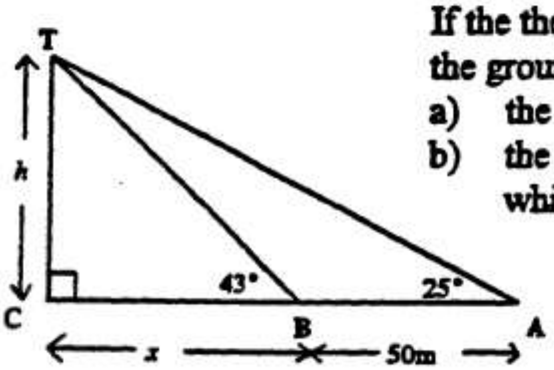
Multiply both sides by 40.

$$BC = 40 \times \tan 70^\circ = 109.8991$$

The boat is 110m. from the base of the cliff.

Example 6

A surveyor using a theodolite measured the angle of elevation of the top of a tower, T, as 25° from point A. He then walked 50 metres towards the tower on horizontal ground and found that the angle of elevation of the top of the tower was 43° from point B.



If the theodolite is 0.75m. vertically above the ground, find the height of,

a) the tower above the ground,
b) the distance x, shown on the diagram, which is not drawn to scale.

a) Using triangle TBC,

$\tan 43° = \dfrac{h}{x}$ so $x = \dfrac{h}{\tan 43°}$ ——— (1)

Using triangle TAC,

$\tan 25° = \dfrac{h}{x + 50}$ so $x\tan 25° + 50\tan 25° = h$

We can now substitute for x from equation (1).

$$\frac{h}{\tan 43°} \times \tan 25° + 50\tan 25° = h$$

Rearranging the formula, we have,

$$h - \frac{h\tan 25°}{\tan 43°} = 50\tan 25°$$

$$h - 0.5000537h = 23.315383$$

$$0.4999462h = 23.315383$$

$$h = \frac{23.315383}{0.4999462}$$

$$h = 46.635778\text{m.}$$

Note that the memory function on a calculator has been used to store values. The final answers are all given from the calculator.

The height of the tower above the ground is

46.635778m. + 0.75m. = 47.385778m. = 47.39m. to two decimal places.

b) From equation (1),

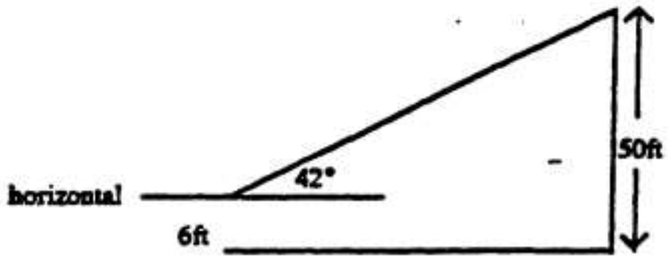
$x = \dfrac{h}{\tan 43°}$ so $x = \dfrac{46.635778}{\tan 43°} = 50.01075$

$x = 50.01$m. to two decimal places.

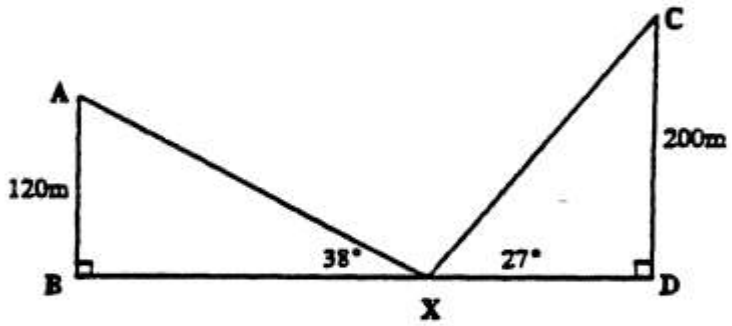
Try this exercise.

Exercise D

1. A man, of height 6 feet, measures the angle of elevation of the top of a 50 feet high tower to be 42°. Calculate to one decimal place how far the man is standing from the tower.

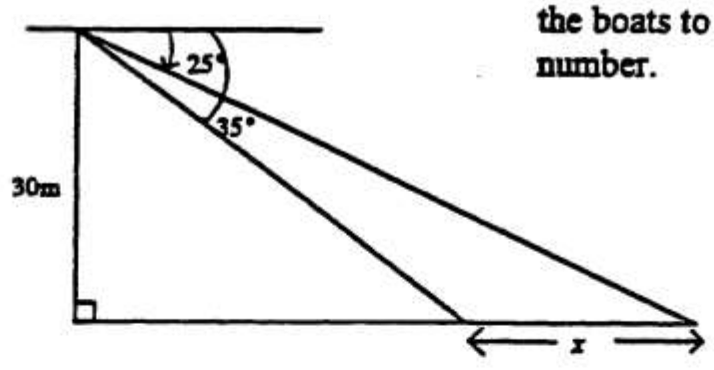


2. The point X is directly between two masts. The mast AB is of height 120m. and the angle of elevation of its top from X is 38°. The mast CD is 200m. high and the angle of elevation of its top from X is 27°. What is the horizontal distance between the two masts, BD, correct to three significant figures?



3. Looking out from the top of a cliff, 30 metres high, two boats can be seen. One is at an angle of depression of 25° and the other is at an angle of depression of 35°.

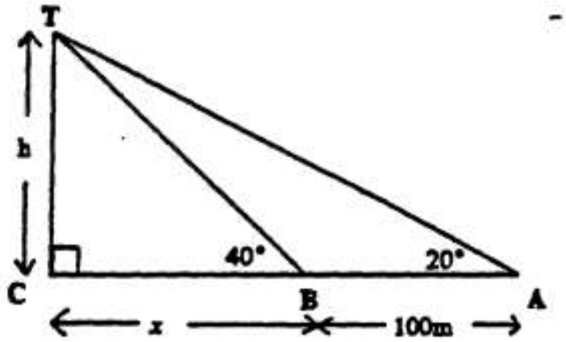
Find the distance x between the boats to the nearest whole number.



Exercise D is continued on the next page.

Exercise D (*Continued*)

4. A surveyor using a theodolite measured the angle of elevation of the top of a tower as 20° from point A. He then walked 100 metres towards the tower on horizontal ground and found that the angle of elevation of the top of the tower was 40° from point B. If the theodolite is 0.75m. vertically above the ground, find correct to one decimal place, the height of,
   a) the tower above the ground,
   b) the distance x on the diagram.



Check your answers with those at the end of the unit.

Three dimensional problems

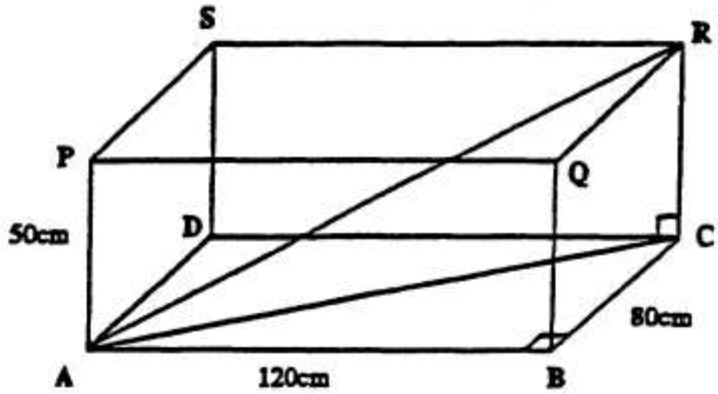
Study these examples. The diagrams are not drawn to scale.

Example 7

The diagram shows a box in the shape of a cuboid ABCDPQRS.
P is vertically above A, Q is vertically above B, R is vertically above C and S is vertically above D. AP = BQ = CR = DS = 50cm.
AB = CD = PQ = RS = 120cm.
BC = AD = QR = PS = 80cm.

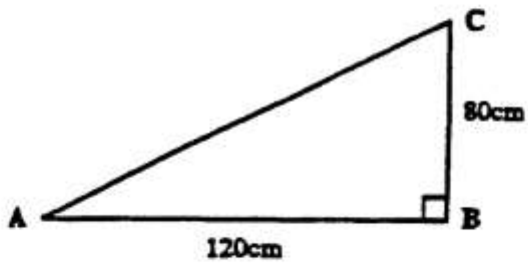
Find, to one decimal place,

a) the length of AC,
b) the length of AR.

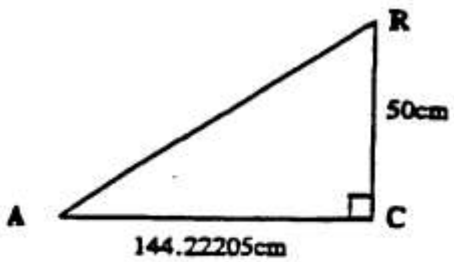


a) To find AC, we use triangle ABC. Angle ABC is a right angle.

We use the theorem of Pythagoras.
$AC^2 = AB^2 + BC^2$
$AC^2 = 120^2 + 80^2$
$AC = 144.22205$
The length of AC is 144.2cm. to one decimal place.



b) To find AR, we use the value found for AC and draw the appropriate right angled triangle. Use Pythagoras as before.
$AR^2 = AC^2 + RC^2$
$AR^2 = 20800 + 2500$
$AR = 152.64338$



The length of AR is 152.6cm. to one decimal place.

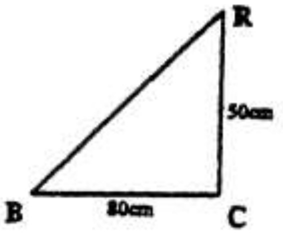
Example 8

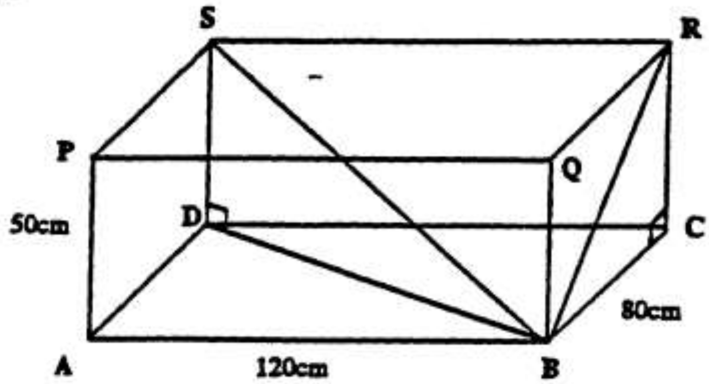
The cuboid shown is the same one used in Example 7. Find, to one decimal place,

a) the angle between the line RB and the plane ABCD,

b) the angle between the line SB and the plane ABCD.

a) Use the right angled triangle RBC.



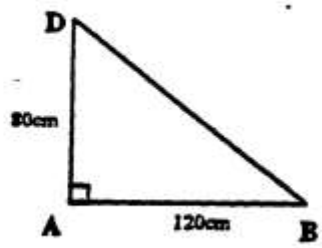


The angle required is angle RBC.

$$\tan RBC = \frac{50}{80}$$

Angle RBC = 32.0° to one decimal place.

b) The angle between the line SB and the plane ABCD is the angle DBS.



Using the theorem of Pythagoras, in the right angled triangle ADB.

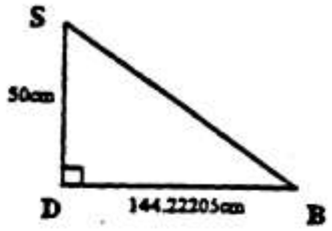
$$DB^2 = AD^2 + AB^2$$

$$DB = 144.22205$$

(It is the same length as AC, which we found in Example 7.)

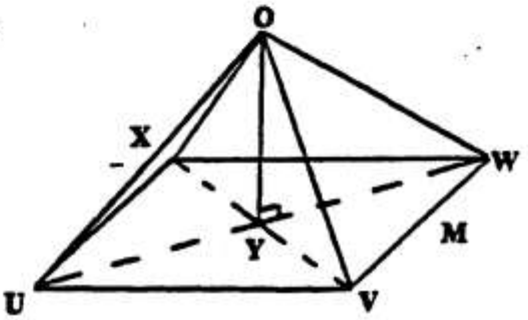
We can now find the angle DBS.

$$\tan DBS = \frac{50}{144.22205}$$

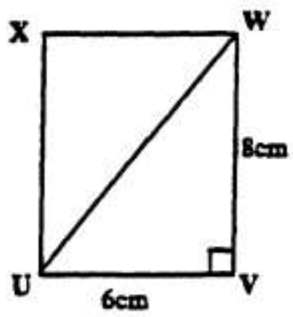


Angle DBS = 19.1° to one decimal place.

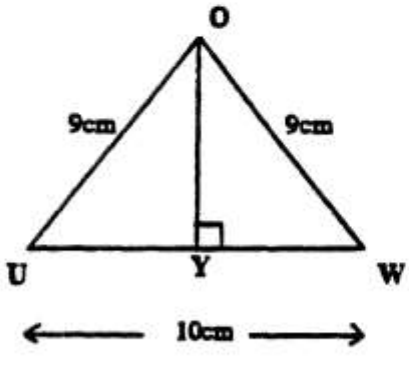
Example 9
OUVWX is a pyramid.
OU = OX = OV = OW = 9cm. VW = UX = 8cm. UV = WX = 6cm.
Find,
a) UW,
b) angle UOW,
c) OY.



a) UVWX is a rectangle.
Using the theorem of Pythagoras,
$UW^2 = UV^2 + VW^2$
$UW^2 = 6^2 + 8^2$
UW = 10cm.



b)



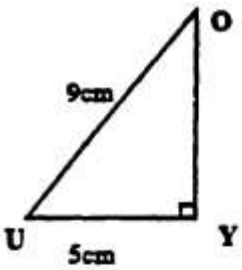
Using the cosine formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos UOW = \frac{9^2 + 9^2 - 10^2}{(2)(9)(9)}$$

Angle UOW = 67.497977°
= 67.5°
to 3 significant figures.

c) We now use triangle OUY.



UY = 0.5 x UW
= 5cm.

Using Pythagoras
$OU^2 = UY^2 + OY^2$
$9^2 = 5^2 + OY^2$
$81 = 25 + OY^2$
$OY = \sqrt{(81 - 25)}$
OY = 7.4833148
OY = 7.48 to 3 significant figures.

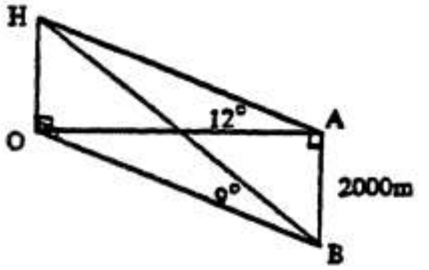
Example 10

From a ship A, due East of a lighthouse, the angle of elevation of the top of the lighthouse is 12°. From a second ship B, which is 2000m. due South of A, the angle of elevation of the top of the lighthouse is 9°.

Find, correct to one decimal place,

a) the height of the lighthouse,

b) the distance of ship B from the lighthouse.



Let OH = h

a) In triangle OHA, $\tan 12° = \dfrac{h}{OA}$ so that $OA = \dfrac{h}{\tan 12°}$

$$OA = 4.7046301h$$

In triangle OHB, $\tan 9° = \dfrac{h}{OB}$ so that $OB = \dfrac{h}{\tan 9°}$

$$OB = 6.3137515h \text{ ——————— } (1)$$

Using Pythagoras in triangle OAB,

$$OB^2 = OA^2 + 2000^2$$
$$(6.3137515h)^2 = (4.7046301h)^2 + 4000000$$
$$39.863458h^2 = 22.133544h^2 + 4000000$$
$$(39.863458 - 22.133544)h^2 = 4000000$$
$$17.729914h^2 = 4000000$$
$$h^2 = \frac{4000000}{17.729914} = 225607.41$$
$$h = 474.98149$$

The height of the lighthouse is 475.0m. to one decimal place.

b) From equation (1)

$$OB = 6.3137515h$$

Using the full value for h,

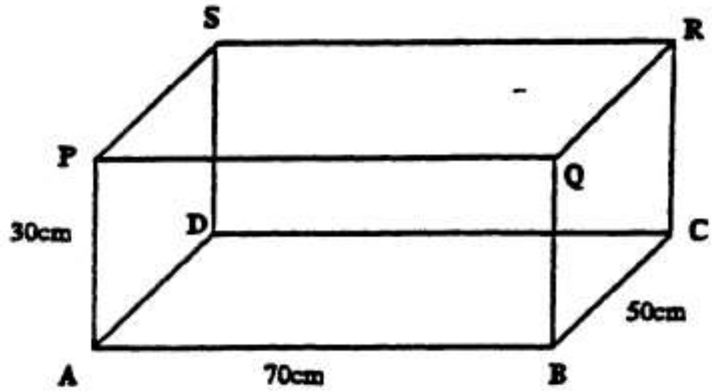
$$OB = 6.3137515 \times 474.98149$$
$$= 2998.9151\text{m.}$$

The distance of ship B from the lighthouse is 2998.9m.

Try this exercise.

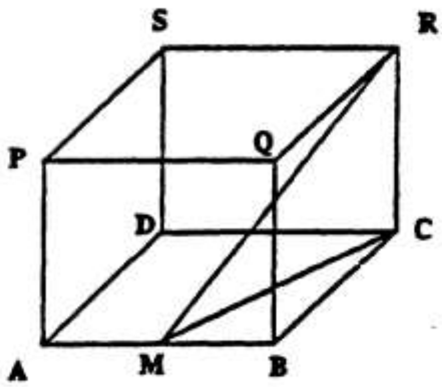
Exercise E

1. The diagram shows a box in the shape of a cuboid ABCDPQRS.
P is vertically above A, Q is vertically above B, R is vertically above C and S is vertically above D.



AP = BQ = CR = DS = 30cm. AB = CD = PQ = RS = 70cm.
BC = AD = QR = PS = 50cm.
Find, to one decimal place,
a) the length of AC,
b) the length of PC.

2.



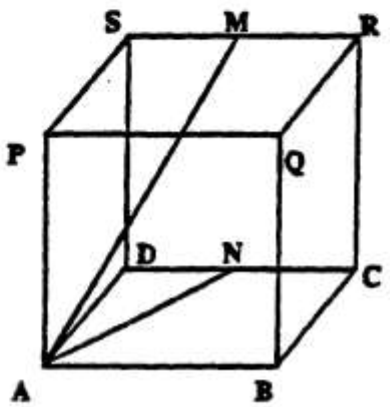
ABCDPQRS is a cube of side 10cm.
AM = MB.

Find, correct to one decimal place,
a) MC,
b) RM,
c) the angle between the line RM and the plane ABCD.

Exercise E is continued on the next page.

Exercise E (*Continued*)

3. ABCDPQRS is a cube of side 12cm. SM = MR = DN = NC.



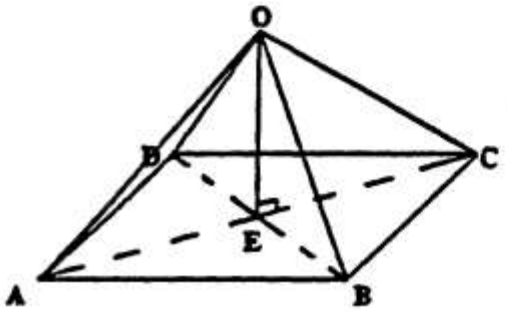
Find, to one decimal place,

a) NA and NB,
b) MA
c) the angle between the line RB and the plane ABCD,
d) the angle between the line MA and the plane ABCD.

4. From a point X, due West of a tower, the angle of elevation of the top of the tower is 18°, but from a point Y, 50m. due North of the tower, the angle of elevation of the top of the tower is 21°. Find, correct to the nearest whole number,
   a) the height of the tower,
   b) the distance between the points X and Y.

5. OABCD is a pyramid standing on a rectangular base, ABCD.
   AB = CD = 4cm. AD = BC = 3cm. OA = OB = OC = OD = 8cm.

Find, correct to one decimal place, where appropriate,



a) the length of AC,
b) angle AOC,
c) the length of OE,
d) the angle OC makes with the plane ABCD.

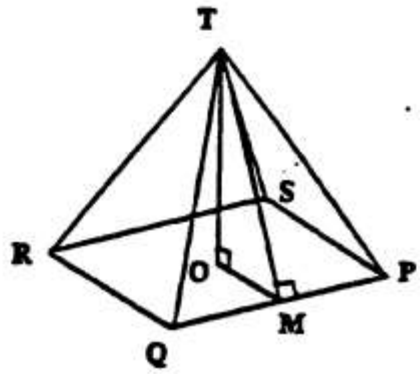
Check your answers with those at the end of the unit.

The angle between two planes

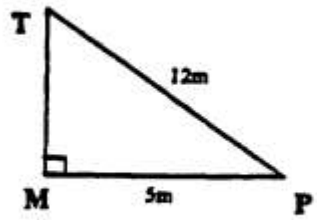
Study Example 11.

Example 11

PQRST is a pyramid. PQRS is a rectangle.
PQ = RS = 10m.   RQ = SP = 8m.
PT = ST = QT = RT = 12m.
T is vertically above the point O.
Find the angle between the planes TQP and PQRS.



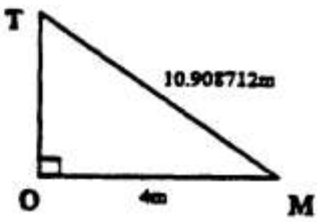
The angle we require is angle TMO.
We need to identify a line where the two planes meet. This line is QP.
Then we find a line in each plane perpendicular to QP. These are OM and TM. Triangle TQP is isosceles (TQ = TP) and therefore if TM is perpendicular to QP then M is the mid-point of QP,
so MP = QP ÷ 2 = 5m.



In triangle TMP, using Pythagoras,

$$TM^2 = 12^2 - 5^2$$
$$= 144 - 25 = 119$$
$$TM = 10.908712\text{m.}$$

O is the centre of the rectangle PQRS
so that OM = RQ ÷ 2 = 4m.



In triangle OTM,

$$\cos TMO = \frac{4}{10.908712} = 0.3666794$$
$$TMO = 68.489028°$$

The angle between the planes TQP and PQRS is 68.5° to one decimal place.

Exercise F is on the next page. You will need to remember that to find the angle between two planes,

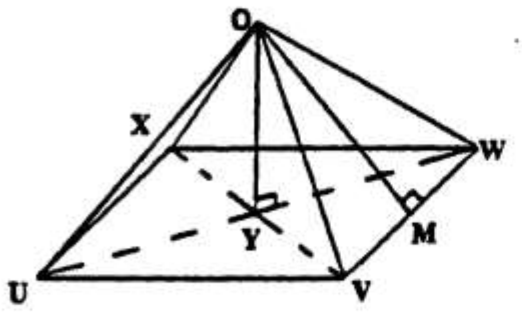
a) find a line where the two planes meet,
b) find a line perpendicular to a) in each plane.

The angle you require is between these two lines.
Remember it is useful to draw appropriate triangles as we did in Example 11.

Try this exercise.

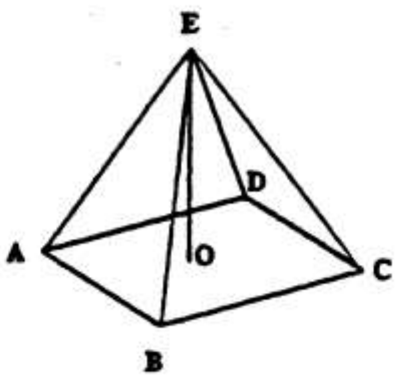
Exercise F

1. The diagram shows a pyramid OUVWX, where UV = WX = 6cm. VW = UX = 10cm. OU = OV = OW = OX = 8cm.
   Find, correct to one decimal place,
   a) the length of OM,
   b) the length of OY,
   c) the angle between the planes OVW and UVWX.



UV = WX = 6cm
VW = UX = 10cm
OU = OV = OW = OX = 8cm

2. The diagram below shows a pyramid standing on a rectangular base ABCD. AB = DC = 5cm. and BC = AD = 7cm. OE is the vertical height of the pyramid and is of length 8cm.
   Find, correct to one decimal place,
   a) the angle between the planes EBC and ABCD,
   b) the angle between the planes ECD and ABCD.



AB = DC = 5cm
BC = AD = 7cm
OE = 8cm

Check your answers with those given at the end of the unit.

Answers

Exercise A
1. 7.8m.
2. 13ft.
3. 17in.
4. 25m.
5. 17.8cm.
6. 2.8km.
7. 10mm.
8. 1.3m.
9. 4.9ft.
10. 6.3cm.

Exercise B
1. 4cm.
2. 6cm.
3. 5cm.
4. 24m.
5. 10m.
6. 12.7mm.
7. 9.8m.
8. 5.7cm.
9. 10.6cm.
10. 13.7m.

Exercise C
1. 17.3ft.
2. a) 40cm.
   b) 240cm.
3. Method 1 80m.
   Method 2 100m.
   Method 3 128.1m.
4. 58.3cm.
5. 11.3ft.

Exercise D
1. 48.9ft.
2. 546m.
3. 21m.
4. a) 65.0m. b) 76.6m.

Exercise E
1. a) 86.0cm. b) 91.1cm.
2. a) 11.2cm. b) 15.0cm.
   c) 41.8°
3. a) NA = 13.4cm. = NB
   b) 18.0cm.
   c) 45.0°
   d) 41.8°
4. a) 19m.
   b) 77m.
5. a) 5cm. b) 36.4°
   c) 7.6cm. d) 71.8°

Exercise F
1. a) 6.2cm.
   b) 5.5cm.
   c) 61.3°
2. a) 72.6°
   b) 66.4°