MATHS Xtra

Unit 20

The sine and cosine rules

Objectives

On completion of this unit you should be able to:

1. Use the sine rule.

2. Understand the ambiguous case.

3. Use the cosine rule.

4. Apply the sine and cosine rule to problems.

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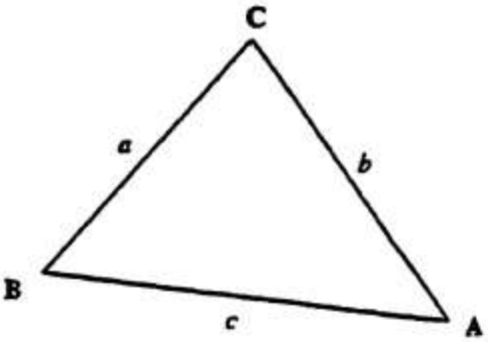
The sine rule

So far, you have been able to find the lengths of all the sides and all the angles of a right angled triangle, if you were given either one angle and one side, or two sides. Now we need to find the angles and sides of a triangle that is not right angled. If we find all the angles and sides of the triangle, it is called solving the triangle.
This triangle is drawn in the usual way.

a is the length of the side opposite angle A,
b is the length of the side opposite angle B and
c is the length of the side opposite angle C.



The sine rule states,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Study this example.

Example 1
Solve the triangle ABC where angle A = 36°, angle B = 64° and a = 5.3m.
Give lengths correct to one decimal place and angles correct to the nearest degree.

Angle A = 36°, angle B = 64°. The three angles of a triangle add up to 180°, so,

angle C = 180° - 36° - 64° = 80°

Using the sine rule,

$$\frac{5.3}{\sin 36^\circ} = \frac{b}{\sin 64^\circ} = \frac{c}{\sin 80^\circ}$$

To find b, we use,

$$\frac{5.3}{\sin 36^\circ} = \frac{b}{\sin 64^\circ}$$

Multiply both sides by sin 64°,

$$b = \frac{5.3 \sin 64^\circ}{\sin 36^\circ}$$

Calculator:
5.3 x 64 sin ÷ 36 sin = 8.1043347
so, b = 8.1m. to one decimal place.

To find c, we use,

$$\frac{5.3}{\sin 36^\circ} = \frac{c}{\sin 80^\circ}$$

Multiply both sides by sin 80°,

$$c = \frac{5.3 \sin 80^\circ}{\sin 36^\circ}$$

Calculator:
5.3 x 80 sin ÷ 36 sin = 8.8799116
c = 8.9m. to one decimal place.

Note that it is always worth checking that the longest side is opposite the largest angle. If it is not you have made a mistake.

Try the following exercise.

Exercise A

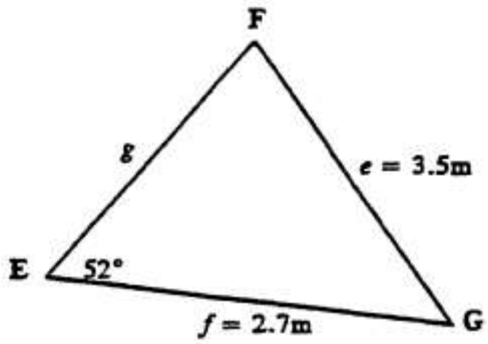
Solve the following triangles, giving all lengths correct to one decimal place.

1. **Triangle ABC, if angle A = 110°, angle B = 15° and a = 8.9cm.**
2. **Triangle RST, if angle R = 82°, angle S = 53° and s = 4.1mm.**
3. **Triangle XYZ, if angle X = 53°, angle Y = 65° and x = 5.2cm.**
4. **Triangle PQR, if angle Q = 75°, angle R = 69° and p = 7.6m.**
5. **Triangle ABC, if angle C = 38°, b = 5.4cm. and angle A = 56°.**

Check your answers to this exercise with those at the end of the booklet.

Example 2

Given that in triangle EFG angle E = 52°, e = 3.5m. and f = 2.7m., find angle F and angle G.



Using the sine rule,

$$\frac{e}{\sin E} = \frac{f}{\sin F} = \frac{g}{\sin G}$$

$$\frac{3.5}{\sin 52°} = \frac{2.7}{\sin F} = \frac{g}{\sin G}$$

Using the first two parts,

$$\sin F = \frac{2.7 \sin 52°}{3.5} = 0.607894$$

angle F = 37.437381° = 37.4° to one decimal place.

There is another possible value for angle F. You should remember from the sine curves that there are several angles whose sine is 0.607894.

Another angle with this value is 180° - 37.4° = 142.6°, to one decimal place.
but e is larger than f therefore angle E must be larger than angle F.

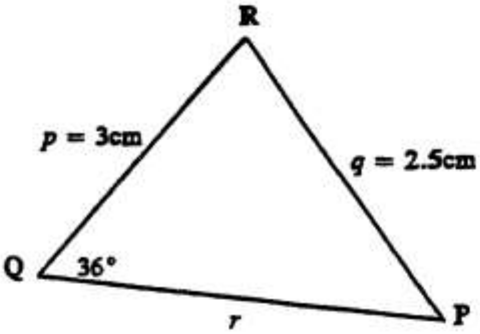
This means that angle F = 37.4°.
Angle G = 180° - 52° - 37.4° = 90.6°.

The ambiguous case

There are times when only one angle and two sides are given and there are two triangles which could result. This is known as the ambiguous case.
This is illustrated in the next example.

Example 3

Solve the triangle PQR where angle Q = 36°, p = 3cm. and q = 2.5cm. Give final answers to one decimal place.



Using the sine rule,

$$\frac{3}{\sin P} = \frac{2.5}{\sin 36°} = \frac{r}{\sin R}$$

Use the first two parts
and transpose to find sin P.

$$\sin P = \frac{3 \sin 36°}{2.5} = 0.7053423.$$

Using your calculator you should find that angle P is equal to 44.86° to two decimal places. There is more than one angle whose sine is 0.7053423. Another angle which has this value for sin P is,

$$180° - 44.86 = 135.14°.$$

Angle P = 44.86° or 135.14°. Either of these angles are possible solutions.

This leads to two different solutions.

Solution 1

If angle P = 44.86°,
angle R = 180° - 36° - 44.86°
= 99.14°

so,

$$\frac{2.5}{\sin 36°} = \frac{r}{\sin 99.14°}$$

$$r = \frac{2.5 \sin 99.14°}{\sin 36°} = 4.2\text{cm. (to 1 d.p.)}$$

angle P = 44.9°
angle Q = 36°
angle R = 99.1°
p = 3cm. q = 2.5cm. r = 4.2cm.

Solution 2

If angle P = 135.14°,
angle R = 180° - 36° - 135.14°
= 8.86°

so,

$$\frac{2.5}{\sin 36°} = \frac{r}{\sin 8.86°}$$

$$r = \frac{2.5 \sin 8.86°}{\sin 36°} = 0.7\text{cm. (to 1 d.p.)}$$

angle P = 135.1°
angle Q = 36°
angle R = 8.9°
p = 3cm. q = 2.5cm. r = 0.7cm.

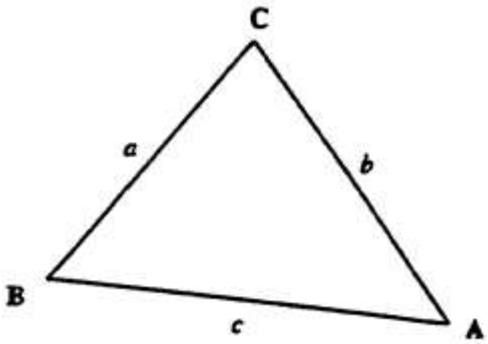
Try this exercise.

Remember to place any answers in the memory of your calculator in order to use the more accurate figure in calculations and not the approximated answer.

Exercise B
Solve the following triangles, giving all answers correct to one decimal place.
1. Triangle PQR if, angle Q = 51°, q = 12m. and r = 8m.
2. Triangle DEF if, d = 75cm., e = 62cm. and angle E = 42°.
3. Triangle ABC if, c = 5mm., b = 6mm and angle C = 27°.
4. Triangle RST if, angle T = 54°, s = 5.2cm. and t = 7.5cm.
5. Triangle IJK if, i = 3.1m., j = 2.3m. and angle J = 35°.

Check your answers to this exercise with those at the end of the booklet.

The cosine rule



The cosine rule states,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

If we transpose this equation for a^2, we find,

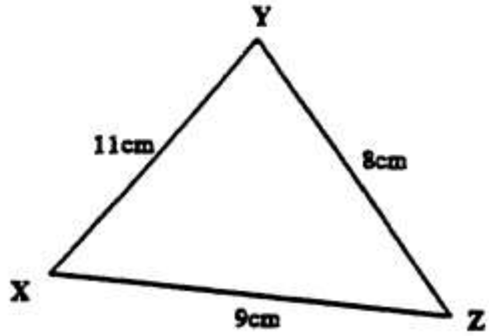
$$a^2 = b^2 + c^2 - 2bc \cos A$$

These two equations can be used when the information given is not sufficient to use the sine rule.

Work through the examples on the next page.

Example 4

In triangle XYZ, if XY = 11cm., YZ = 8cm. and XZ = 9cm, find angle Z to one decimal place.



$$\cos Z = \frac{XZ^2 + YZ^2 - XY^2}{2(XZ)(YZ)}$$

$$= \frac{81 + 64 - 121}{2 \times 9 \times 8}$$

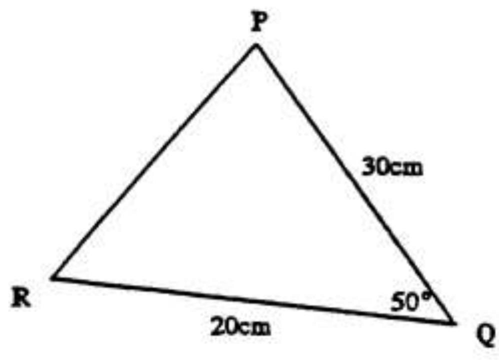
$$= \frac{24}{144}$$

$$= 0.1666667$$

angle Z = 80.4°.

Example 5

Triangle PQR has PQ = 30cm., RQ = 20cm. and angle Q = 50°. Find PR.



Using the cosine rule,

$PR^2 = PQ^2 + RQ^2 - 2(PQ)(RQ)\cos 50°$

$PR^2 = 900 + 400 - 2 \times 30 \times 20 \times \cos 50°$

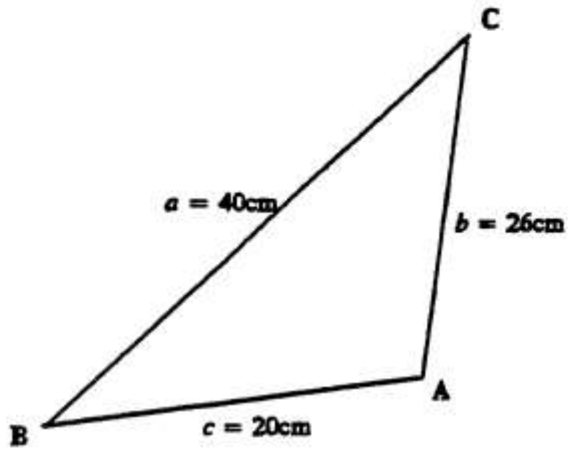
$PR^2 = 528.65487$

$PR = 22.992496$

PR is equal to 23.0cm. to one decimal place.

Example 6

Triangle ABC has a = 40cm., b = 26cm. and c = 20cm. Find angle A.



Using the cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{26^2 + 20^2 - 40^2}{2 \times 26 \times 20}$$

$$\cos A = \frac{-524}{1040}$$

$$= -0.5038461$$

Angle A = 120.3°. Note that the cosine of angle A is negative. The trig graphs showed that for angles between 90° and 270° the cosine of the angle is negative. The angle we require is obtuse and is therefore equal to 120.3°.

Note: If a triangle is to be solved and the three sides are given, always use the cosine rule to determine the largest angle first as this is the only angle which could be obtuse and the cosine rule would show this. (The largest angle is opposite the largest side.)

Now try the exercise on the next page.

Exercise C

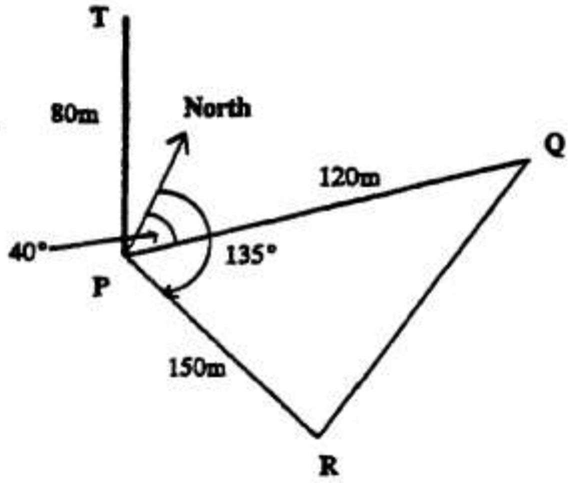
1. Triangle ABC has sides AB = 3.6cm., BC = 4.7cm. and AC = 5.3cm. Find the size of angle B to the nearest degree.
2. In triangle PQR, p = 15mm., q = 17mm. and r = 28mm., find the size of the largest angle correct to one decimal place.
3. In triangle RST, RT = 42cm., ST = 57cm. and angle T = 62°. Find the length of RS, correct to two decimal places.
4. In triangle XYZ, angle X = 103°, y = 3.9m. and z = 4.7m. Find the length of x correct to one decimal place.
5. Find the three angles of the triangle ABC, if AB = 8mm., AC = 13mm. and BC = 7mm. Give all answers correct to one decimal place.
6. Solve the triangle PQR, if angle Q is 130°, QP = 4.6cm. and QR = 7.3cm. Give all answers correct to one decimal place.

Check your answers with those given at the end of the booklet before continuing to the next example.

Study this example of a practical problem.

Example 7

PT is a vertical tower 80m. high. Q is a point on the land 120m. horizontally from the base of the tower on a bearing of 040°. R is another point on a bearing of 135° and 150m. from the base of the tower.



Find,

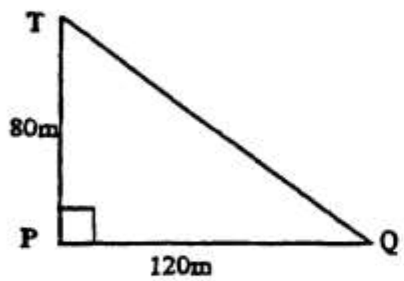
a) the distance between Q and R,

b) the angle of elevation of the top of the tower, T from Q.

a) The angle QPR is equal to 135° - 40° = 95°.
We shall use the cosine rule on triangle PQR to find QR.

$$QR^2 = PR^2 + PQ^2 - 2(PR)(PQ)\cos 95^\circ$$
$$QR^2 = 150^2 + 120^2 - 2 \times 150 \times 120 \times \cos 95^\circ$$
$$QR^2 = 40037.607$$
$$QR = 200.09399\text{m.} = 200.1\text{m. to one decimal place.}$$

b) The angle required is angle PQT,
this is the angle of elevation of T from Q.



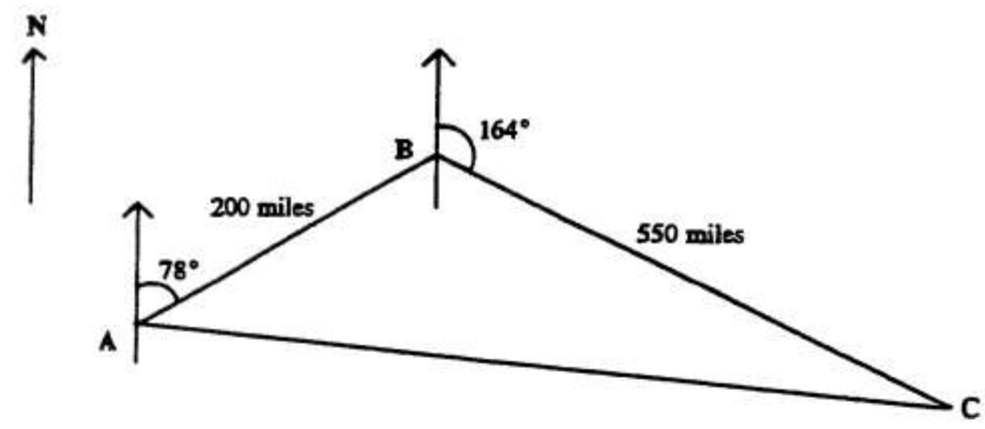
$$\tan PQT = \frac{80}{120}$$

Angle PQT is 33.7° to one decimal place.

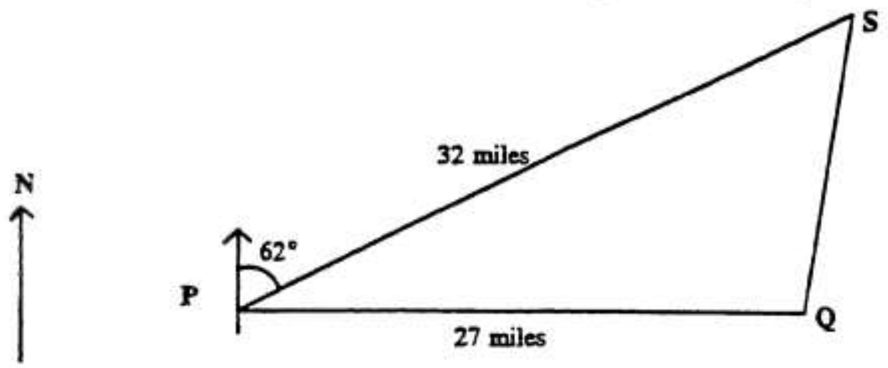
Try the final exercise on the next page.

Exercise D

1. The diagram below, which is not drawn to scale, shows the horizontal path taken by an aircraft, which commences at A and flies 200 miles, on a bearing of 078°, to point B. At B it changes direction and flies 550 miles on a bearing of 164° to point C. Find correct to one decimal place,
   a) the size of angle ABC,
   b) the distance of point A from point C,
   c) the bearing of C from A.



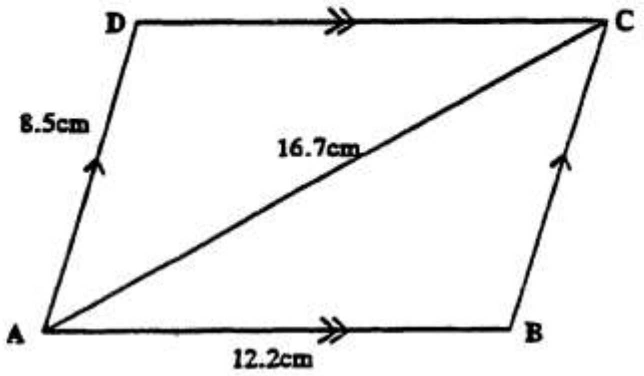
2. The diagram below, which is not drawn to scale, shows the position of two lighthouses P and Q, which are 27 miles apart, with Q directly East of P. A ship, S, is spotted 32 miles from P, on a bearing of 062°. Find,
   a) to one decimal place, the distance of the ship S from the lighthouse Q,
   b) to the nearest degree, the bearing of S from Q.



Exercise D is continued on the next page.

Exercise D (*Continued*)

3. The diagram below shows a parallelogram ABCD, with AB = DC = 12.2cm. and BC = AD = 8 .5cm. The diagonal AC is of length 16.7cm. Find the angles of the parallelogram to the nearest degree.



Check your answers with those given at the end of the unit.

Answers

Exercise A

1. A = 110° B = 15° C = 55°
   a = 8.9cm. b = 2.5cm. c = 7.8cm.
2. R = 82° S = 53° T = 45°
   r = 5.1mm. s = 4.1mm. t = 3.6mm.
3. X = 53° Y = 65° Z = 62°
   x = 5.2cm. y = 5.9cm. z = 5.7cm.
4. P = 36° Q = 75° R = 69°
   p = 7.6m. q = 12.5m. r = 12.1m.
5. A = 56° B = 86° C = 38°
   a = 4.5cm. b = 5.4cm. c = 3.3cm.

Exercise B

1. Angle Q = 51°, q = 12m. and r = 8m.
   Angle R = 31.2°
   Angle P = 97.8°
   p = 15.3m.
   Note: this is not the ambiguous case as angle R has to be smaller than angle Q.

2. Angle E = 42°, d = 75cm. and e = 62cm.

Solution 1	Solution 2
Angle D = 54.0°	Angle D = 126.0°
Angle F = 84.0°	Angle F = 12.0°
f = 92.1cm.	f = 19.3cm.

3. Angle C = 27°, b = 6mm. and c = 5mm.

Solution 1	Solution 2
Angle B = 33.0°	Angle B = 147.0°
Angle A = 120.0°	Angle A = 6.0°
a = 9.5mm.	a = 1.2mm.

4. Angle T = 54°, s = 5.2cm. and t = 7.5cm.
   Angle S = 34.1°
   Angle R = 91.9°
   r = 9.3cm.
   This is not an ambiguous case as angle S must be smaller than angle T.

5. Angle J = 35°, j = 2.3m. and i = 3.1m.

Solution 1	Solution 2
Angle I = 50.6°	Angle I = 129.4°
Angle K = 94.4°	Angle K = 15.6°
k = 4.0m.	k = 1.1m.

Exercise C

1. Angle B = 78°
2. The largest angle = R = 122.0°
3. RS = 52.58cm.
4. x = 6.7m.
5. Angle A = 27.8°
   Angle B = 120.0°
   Angle C = 32.2°
6. PR = 10.8cm.
   Angle P = 31.0°
   Angle R = 19.0°

Exercise D

1. a) 94.0°
   b) 598.2 miles
   c) 144.5°
2. a) 15.1 miles
   b) 005°
3. Angle A = angle C = 74°
   Angle B = angle D = 106°