Unit 14

Algebraic fractions

Objectives

On completion of this unit you should understand:

1. Multiplication and division of algebraic fractions.

2. Simplification of algebraic expressions.

3. Addition and subtraction of algebraic fractions.

Cook, M. and Rimmer, H. (1993),
Stafford College, Earl Street, Stafford, ST16 2QR

Multiplication of fractions

Consider the rules we follow when we multiply two fractions.
Study this example.

Example 1
Evaluate the following.

a) $\frac{2}{3} \times \frac{7}{5}$

b) $\frac{3}{14} \times \frac{7}{9}$

a) For this we simply multiply the numerators and multiply the denominators.

$$\frac{2}{3} \times \frac{7}{5} = \frac{2 \times 7}{3 \times 5} = \frac{14}{15}.$$

b) In this case we can 'cancel'. Both the 3 and the 9 will divide by 3 so we can divide the numerator and the denominator by 3.
Similarly, both the 7 and the 14 will divide by 7 so, we divide both the numerator and the denominator by 7. We must do the same to the numerator as the denominator in each case.

$$\frac{{}^{1}\cancel{3}}{{}_{2}\cancel{14}} \times \frac{\cancel{7}^{1}}{\cancel{9}_{3}} = \frac{1 \times 1}{2 \times 3} = \frac{1}{6}.$$

These rules apply when we multiply algebraic fractions.
Consider this example.

Example 2
Simplify $\frac{12a}{b^2} \times \frac{b}{18a^3}$.

Both the numerator and the denominator will divide by a. Similarly, both the numerator and the denominator will divide by 6 and b.

$$\frac{{}^{2}\cancel{12a}^{1}}{{}_{b}\cancel{b^2}} \times \frac{\cancel{b}^{1}}{{}_{3}\cancel{18a^3}_{a \times a}} = \frac{2 \times 1 \times 1}{b \times 3 \times a \times a} = \frac{2}{3a^2b}.$$

Try the exercise on the next page.

Exercise A

Simplify the following expressions.

1. $\frac{12x^2}{6y^2} \times \frac{9y^3}{x^4}$
2. $\frac{7x^3}{21y^5} \times \frac{30y^3}{40x^2}$
3. $\frac{3cd^2}{6e^2} \times \frac{9e^3}{d^4}$
4. $\frac{12x}{22y^2} \times \frac{11y^3}{4x^4}$
5. $\frac{16xy^3}{48y^2} \times \frac{24xy^5}{8x^8}$
6. $\frac{20x^2}{35xy^2} \times \frac{4y^2}{x^6}$
7. $\frac{18r^2}{6p^2} \times \frac{9p^3}{q^4}$
8. $\frac{12s^2}{15t^7} \times \frac{10t^2}{s^4}$
9. $\frac{8p^5}{5q^2} \times \frac{15qr^5}{40p^8}$
10. $\frac{16x^5}{20y^7} \times \frac{10xy^3}{x^4}$

Check your answers with those at the end of the unit.

Division of fractions

First consider what happens with an arithmetic example, then apply the same rules to an algebraic example.

Example 3

Evaluate $\frac{3}{35} \div \frac{9}{14}$.

To complete this, the rule for division is to change the ÷ to x and turn the second fraction upside down.

$$\frac{3}{35} \times \frac{14}{9}$$

Divide numerator and denominator by 3 and 7.

$$= \frac{1 \times 2}{5 \times 3} = \frac{2}{15}.$$

Example 4

Simpify $\frac{48y^2}{20x^7} \div \frac{6y^3}{x^4}$.

$$\frac{48y^2}{20x^7} \div \frac{6y^3}{x^4} = \frac{48y^2}{20x^7} \times \frac{x^4}{6y^3}$$

Cancel as before or write out in full and then cancel.

$$= \frac{48 \times \cancel{y} \times \cancel{y} \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x}}{20 \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x} \times x \times x \times x \times 6 \times \cancel{y} \times \cancel{y} \times y} = \frac{48}{120x^3y} = \frac{2}{5x^3y}.$$

Try this exercise.

Exercise B
Simplify the following expressions.

1. $\dfrac{12x^2}{5y^2} \div \dfrac{9x^3}{y^2}$
2. $\dfrac{7x^3}{21y^5} \div \dfrac{30x^2}{4y^2}$
3. $\dfrac{5cd^4}{6e^2} \div \dfrac{10d^3}{e^5}$
4. $\dfrac{12yz^2}{22x^2} \div \dfrac{24xz^3}{4y^4}$
5. $\dfrac{16xyz^3}{48y^2} \div \dfrac{24yz^5}{8x^2}$
6. $\dfrac{2xy^2}{35z^2} \div \dfrac{4y^2}{x^4}$
7. $\dfrac{18r^2}{6p^2} \div \dfrac{9r^3}{q^4}$
8. $\dfrac{12s^2}{15t^7} \div \dfrac{10s^2}{t^4}$
9. $\dfrac{8pr^4}{5q^2} \div \dfrac{15pr^5}{40q^8}$
10. $\dfrac{16x^5}{20y^7} \div \dfrac{32x^5}{80y^7}$

Check your answers with those at the end of the unit.

Simplification of expressions

Study this example.

Example 5
Simplify the following expressions.

a) $\dfrac{x - 3}{4x - 12}$

b) $\dfrac{xy - 3x^2}{5xy - 15x^2}$

c) $\dfrac{x - 3}{x^2 + 2x - 15}$

d) $\dfrac{x^2 + x - 6}{x^2 + 2x - 8}$

For each of these examples we factorise and cancel, where appropriate.

a) $\dfrac{x - 3}{4x - 12} = \dfrac{\cancel{(x - 3)}}{4\cancel{(x - 3)}} = \dfrac{1}{4}.$

b) $\dfrac{xy - 3x^2}{5xy - 15x^2} = \dfrac{\cancel{x}\cancel{(y - 3x)}}{5\cancel{x}\cancel{(y - 3x)}} = \dfrac{1}{5}.$

c) $\dfrac{x - 3}{x^2 + 2x - 15} = \dfrac{\cancel{(x - 3)}}{\cancel{(x - 3)}(x + 5)} = \dfrac{1}{(x + 5)}.$

d) $\dfrac{x^2 + x - 6}{x^2 + 2x - 8} = \dfrac{\cancel{(x - 2)}(x + 3)}{\cancel{(x - 2)}(x + 4)} = \dfrac{(x + 3)}{(x + 4)}.$

Try this exercise.

Exercise C

Simplify the following expressions.

1. $\dfrac{x+2}{3x+6}$
2. $\dfrac{2x-10}{x-5}$
3. $\dfrac{y-3x^2}{5y-15x^2}$
4. $\dfrac{2p-8p^2}{3p^2-12p^3}$
5. $\dfrac{x-1}{x^2-2x+1}$
6. $\dfrac{x-9}{x^2-8x-9}$
7. $\dfrac{x+3}{x^2-x-12}$
8. $\dfrac{x^2+x-2}{x^2+2x-3}$
9. $\dfrac{x^2+6x+5}{x^2+7x+10}$
10. $\dfrac{x^2-8x+7}{x^2-5x-14}$

Check your answers with those at the end of the unit.

Addition and subtraction of fractions

We shall again use an arithmetic example, followed by an algebraic one.
The rules are,

find a common denominator,
express each fraction using this denominator,
add or subtract the numerators.

Consider these examples.

Example 6

Evaluate $\dfrac{4}{5} + \dfrac{11}{20} - \dfrac{7}{15}$.

To find the lowest common denominator of 5, 20 and 15 start with the largest of the three numbers, 20.
5 will divide exactly into 20 but 15 will not.
Next try 20 x 2 = 40. Again 5 will divide exactly into 40 but 15 will not.
Try 20 x 3 = 60. This time 5 divides into 60 twelve times, 15 divides into 60 four times, and 20 divides into 60 three times, so 60 is the lowest common denominator of the three numbers.

$$\frac{4}{5} + \frac{11}{20} - \frac{7}{15} = \frac{4 \times 12}{5 \times 12} + \frac{11 \times 3}{20 \times 3} - \frac{7 \times 4}{15 \times 4} = \frac{48}{60} + \frac{33}{60} - \frac{28}{60} = \frac{53}{60}.$$

Example 7

Simplify $\frac{x}{3} - \frac{5x}{6} + \frac{7x}{8}$.

The lowest common denominator is 24 so we can continue.

$$\frac{x}{3} - \frac{5x}{6} + \frac{7x}{8} = \frac{x \times 8}{3 \times 8} - \frac{5x \times 4}{6 \times 4} + \frac{7x \times 3}{8 \times 3} = \frac{8x - 20x + 21x}{24} = \frac{9x}{24} = \frac{3x}{8}.$$

Example 8

Simplify $\frac{2}{3x} - \frac{5}{6x} + \frac{7}{9x^2}$.

The lowest common denominator is $18x^2$.

$18x^2 \div 3x = 6x$ $\quad 18x^2 \div 6x = 3x$ $\quad 18x^2 \div 9x^2 = 2$

We can now continue.

$$\frac{2}{3x} - \frac{5}{6x} + \frac{7}{9x^2} = \frac{2 \times 6x}{3x \times 6x} - \frac{5 \times 3x}{6x \times 3x} + \frac{7 \times 2}{9x^2 \times 2}$$

$$= \frac{2 \times 6x}{18x^2} - \frac{5 \times 3x}{18x^2} + \frac{7 \times 2}{18x^2}$$

$$= \frac{12x - 15x + 14}{18x^2}$$

$$= \frac{-3x + 14}{18x^2}.$$

Try this exercise.

Exercise D

Simplify the following expressions.

1. $\frac{a}{12} - \frac{7a}{36}$
2. $\frac{5a}{12} + \frac{11a}{18}$
3. $\frac{20xy}{25} - \frac{8xy}{50}$
4. $\frac{4}{5p} + \frac{5}{12p}$
5. $\frac{5a}{12b} - \frac{b}{9a}$
6. $\frac{3}{4y} + \frac{5}{6y} - \frac{7}{12y}$
7. $\frac{1}{4y} - \frac{5x}{24y} - \frac{7}{12y}$
8. $\frac{3x}{5y} + \frac{5x}{10y} - \frac{7}{30y}$
9. $\frac{1}{4z} + \frac{3}{8z^2} - \frac{5}{16z}$
10. $\frac{3}{4p} + \frac{7}{12p} - \frac{7}{12p^2}$

Check your answers with those at the end of the unit.

Now study this example.

Example 9

Express as a single fraction and simplify $\frac{p}{12} + \frac{(2p+q)}{4} - \frac{(p-2q)}{6}$.

The lowest common denominator is 12.

The first term already has 12 as a denominator, so both the numerator and the denominator remain unchanged. We need to multiply the numerator and denominator of the second term by 3 and the numerator and denominator of the third term need to be multiplied by 2.

$$\frac{p}{12} + \frac{(2p+q)}{4} - \frac{(p-2q)}{6} = \frac{p}{12} + \frac{3(2p+q)}{4 \times 3} - \frac{2(p-2q)}{6 \times 2}$$

$$= \frac{p}{12} + \frac{3(2p+q)}{12} - \frac{2(p-2q)}{12}$$

Multiply the brackets out.

$$= \frac{p + 6p + 3q - 2p + 4q}{12}$$

Collect like terms.

$$= \frac{5p + 7q}{12}.$$

Try this exercise.

Exercise E

1. $\frac{x}{15} + \frac{(2x-3y)}{10} - \frac{(x-2y)}{5}$
2. $\frac{2(x+4)}{5} + \frac{(3x-9y)}{20} + \frac{(x-9)}{5}$
3. $\frac{3x}{4} - \frac{(5x-2y)}{12} - \frac{(3x+8y)}{8}$
4. $\frac{b}{3} + \frac{(5a-7b)}{4} - \frac{(2a-3b)}{3}$
5. $\frac{x}{15} + \frac{(x-3y)}{20} - \frac{(x-y)}{5}$
6. $\frac{p}{15} + \frac{(2q-3p)}{10} - \frac{(10q-2p)}{6}$

Check your answers with those at the end of the unit.

Study these examples.

Example 10

Simplify $\frac{2}{(x+1)} + \frac{3}{(x-1)}$.

As before, we need to find a common denominator. This will be $(x + 1)(x - 1)$.

$$\frac{2}{(x+1)} + \frac{3}{(x-1)} = \frac{2 \text{ x } (x-1)}{(x+1)(x-1)} + \frac{3 \text{ x } (x+1)}{(x-1)(x+1)}$$

The denominator of the first term has been multiplied by $(x - 1)$ so we do the same to the numerator. Similarly, the denominator of the second term has been multiplied by $(x + 1)$, so we multiply this numerator by $(x + 1)$.

We can now write this over a common denominator.

$$= \frac{2(x-1) + 3(x+1)}{(x+1)(x-1)}$$

Multiply out the brackets.

$$= \frac{2x - 2 + 3x + 3}{(x+1)(x-1)}$$

$$= \frac{5x+1}{(x+1)(x-1)}.$$

Example 11

Simplify $\frac{4}{(x+2)} + \frac{3}{(x-5)} + \frac{3}{x}$.

The common denominator is $x(x + 2)(x - 5)$.

The numerator and denominator of the first term will be multiplied by $x(x - 5)$.

The numerator and denominator of the second term will be multiplied by $x(x + 2)$.

The numerator and denominator of the third term will be multiplied by $(x + 2)(x - 5)$.

$$\frac{4}{(x+2)} + \frac{3}{(x-5)} + \frac{3}{x} = \frac{4x(x-5)}{x(x+2)(x-5)} + \frac{3x(x+2)}{x(x+2)(x-5)} + \frac{3(x+2)(x-5)}{x(x+2)(x-5)}$$

We put these terms over the same common denominator and multiply out the brackets.

$$= \frac{4x^2 - 20x + 3x^2 + 6x + 3(x^2 - 5x + 2x - 10)}{x(x+2)(x-5)}$$

$$= \frac{4x^2 - 20x + 3x^2 + 6x + 3x^2 - 15x + 6x - 30}{x(x+2)(x-5)}$$

$$= \frac{10x^2 - 23x - 30}{x(x+2)(x-5)}.$$

Try this exercise.

Exercise F

Simplify the following expressions.

1. $\dfrac{4}{(x+6)} + \dfrac{5}{(x-3)}$

2. $\dfrac{1}{(x-6)} + \dfrac{2}{(x-2)}$

3. $\dfrac{3}{(x-4)} + \dfrac{1}{(2x-3)}$

4. $\dfrac{1}{(2x+6)} + \dfrac{5}{(x-2)}$

5. $\dfrac{4}{(3x+3)} + \dfrac{1}{(x-5)}$

6. $\dfrac{2}{(x+4)} + \dfrac{3}{5x} - \dfrac{1}{(x-1)}$

7. $\dfrac{1}{(2x+3)} + \dfrac{2}{7x} + \dfrac{1}{(x+1)}$

8. $\dfrac{4}{(x+2)} - \dfrac{1}{x} - \dfrac{1}{(2x-3)}$

9. $\dfrac{1}{(3x-1)} - \dfrac{3}{x} + \dfrac{2}{(x+1)}$

10. $\dfrac{2}{(x+1)} + \dfrac{2}{x} - \dfrac{1}{(x-1)}$

Check your answers with those at the end of the unit.

Answers

Exercise A

1. $\frac{18y}{x^2}$
2. $\frac{x}{4y^2}$
3. $\frac{9ce}{2d^2}$
4. $\frac{3y}{2x^3}$
5. $\frac{y^6}{x^6}$
6. $\frac{16}{7x^5}$
7. $\frac{27pr^2}{q^4}$
8. $\frac{8}{s^2t^5}$
9. $\frac{3r^5}{5p^3q}$
10. $\frac{8x^2}{y^4}$

Exercise B

1. $\frac{4}{15x}$
2. $\frac{2x}{45y^3}$
3. $\frac{cde^3}{12}$
4. $\frac{y^5}{11x^3z}$
5. $\frac{x^3}{9y^2z^2}$
6. $\frac{x^5}{70z^2}$
7. $\frac{q^4}{3p^2r}$
8. $\frac{2}{25t^3}$
9. $\frac{64q^6}{15r}$
10. 2

Exercise C

1. $\frac{1}{3}$
2. 2
3. $\frac{1}{5}$
4. $\frac{2}{3p}$
5. $\frac{1}{(x-1)}$
6. $\frac{1}{(x+1)}$
7. $\frac{1}{(x-4)}$
8. $\frac{(x+2)}{(x+3)}$
9. $\frac{(x+1)}{(x+2)}$
10. $\frac{(x-1)}{(x+2)}$

Exercise D

1. $\frac{-a}{9}$
2. $\frac{37a}{36}$
3. $\frac{16xy}{25}$
4. $\frac{73}{60p}$
5. $\frac{15a^2 - 4b^2}{36ab}$
6. $\frac{1}{y}$
7. $\frac{-8-5x}{24y}$
8. $\frac{33x-7}{30y}$
9. $\frac{-z+6}{16z^2}$
10. $\frac{16p-7}{12p^2}$

Exercise E

1. $\frac{2x+3y}{30}$
2. $\frac{15x-9y-4}{20}$
3. $\frac{-x-20y}{24}$
4. $\frac{7a-5b}{12}$
5. $\frac{-5x+3y}{60}$
6. $\frac{3p-44q}{30}$

Exercise F

1. $\frac{9(x+2)}{(x+6)(x-3)}$
2. $\frac{3x-14}{(x-6)(x-2)}$
3. $\frac{7x-13}{(x-4)(2x-3)}$
4. $\frac{11x+28}{(2x+6)(x-2)} = \frac{11x+28}{2(x+3)(x-2)}$
5. $\frac{7x-17}{(3x+3)(x-5)} = \frac{7x-17}{3(x+1)(x-5)}$
6. $\frac{8x^2-21x-12}{5x(x+4)(x-1)}$
7. $\frac{25^2+38x+6}{7x(2x+3)(x+1)}$
8. $\frac{5x^2-15x+6}{x(x+2)(2x-3)}$
9. $\frac{-2x^2-7x+3}{x(3x-1)(x+1)}$
10. $\frac{3x^2-3x-2}{x(x+1)(x-1)}$