MATHS Xtra
ALGEBRA

Unit 13

Binomial Series

Objectives

On completion of this unit you should be able to use:

1. The binomial series.

2. The binomial approximation for small errors.

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The binomial series

You have already used the binomial theorem.
The theorem is as follows.

$$(a+b)^n = a^n + \frac{n a^{(n-1)} b}{1!} + \frac{n(n-1) a^{(n-2)} b^2}{2!} + \frac{n(n-1)(n-2) a^{(n-3)} b^3}{3!} + \dots\dots\dots + b^n$$

If we now put a = 1 and b = x then we obtain the binomial series, as follows.

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots\dots\dots + x^n$$

This is true for all positive whole number values for n.

For negative and fractional values of n, there is no final term on the right hand side of this equation, so we have,

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots\dots\dots\dots\dots$$

If $-1 < x < 1$ then $|x^2|$ is larger than $|x^3|$ and $|x^3|$ is larger than $|x^4|$ and so on. The series is said to converge as the power of x becomes higher the value of the term involved becomes less significant. The binomial series is only relevant if $|x| < 1$, and when n is fractional or negative.

Consider the following examples.

Example 1

Use the binomial series to expand $(1 - 3x)^{-1}$ to four terms.

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots\dots\dots\dots\dots$$

We need to let $x \equiv (-3x)$ and n = -1.

$$(1-3x)^{-1} = 1 + \frac{(-1)(-3x)}{1!} + \frac{(-1)(-2)(-3x)^2}{2!} + \frac{(-1)(-2)(-3)(-3x)^3}{3!} + \dots\dots\dots$$

$$(1-3x)^{-1} = 1 + \frac{3x}{1!} + \frac{18x^2}{2!} + \frac{162x^3}{3!} + \dots\dots\dots\dots\dots$$

$$(1-3x)^{-1} = 1 + 3x + 9x^2 + 27x^3 + \dots\dots\dots\dots\dots$$

Example 2
Use the binomial series to expand $(1 + 2x)^{-1/2}$ to four terms.

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots\ldots\ldots\ldots$$

We need to let $x \equiv (2x)$ and $n = {}^{-1}/_{2}$.

$$(1 + 2x)^{-1/2} = 1 + \frac{(-\frac{1}{2})(2x)}{1!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(2x)^2}{2!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(2x)^3}{3!} + \ldots\ldots$$

$$(1 + 2x)^{-1/2} = 1 - x + \frac{3x^2}{2} - \frac{5x^3}{2} + \ldots\ldots\ldots\ldots$$

Example 3
Use the binomial series to expand $\sqrt{(1 - 4x)}$ to four terms.

$\sqrt{(1 - 4x)}$ is the same as $(1 - 4x)^{1/2}$.

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots\ldots\ldots\ldots$$

We need to let $x \equiv (-4x)$ and $n = {}^{1}/_{2}$.

$$(1 - 4x)^{1/2} = 1 + \frac{(\frac{1}{2})(-4x)}{1!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-4x)^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-4x)^3}{3!} + \ldots\ldots$$

$$(1 - 4x)^{1/2} = 1 - 2x - 2x^2 - 4x^3 + \ldots\ldots\ldots\ldots$$

Try this exercise.

Exercise A
Use the binomial series to expand each of the following to four terms.

1. $(1 + x)^{-2}$
2. $(1 - x)^{1/2}$
3. $(1 + 5y)^{-1}$
4. $(1 - 2y)^{-3}$
5. $\sqrt{(1 - 2x)}$
6. $\dfrac{1}{(1 - 3x)^{1/2}}$

   Hint: $\dfrac{1}{(1 - 3x)^{1/2}} = (1 - 3x)^{-1/2}$
7. $(1 + 4x)^{-4}$
8. $(1 + 2x)^{-1}$
9. $(1 - x)^{-2}$
10. $(1 + 2x)^{-2}$
11. $\sqrt{(1 + x)}$
12. $\dfrac{1}{(1 - 3x)^{1/3}}$

Check your answers with those at the end of the unit.

Convergence

Consider these examples.

Example 4

Using Example 2, where we used the binomial series to expand $(1 + 2x)^{-1/2}$ to four terms. Find the value of the series if $x = 0.01$.

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots\ldots\ldots$$

We let $x \equiv (2x)$ and $n = -\frac{1}{2}$ and obtained,

$$(1 + 2x)^{-1/2} = 1 - x + \frac{3x^2}{2} - \frac{5x^3}{2} + \ldots\ldots\ldots$$

We now let $x = 0.01$.

$$[1 + 2(0.01)]^{-1/2} = 1 - (0.01) + \frac{3(0.01)^2}{2} - \frac{5(0.01)^3}{2} + \ldots\ldots\ldots$$

$$[1 + 2(0.01)]^{-1/2} = 1 - 0.01 + 0.00015 - 0.0000025 + \ldots\ldots\ldots$$

$$[1 + 2(0.01)]^{-1/2} \approx 0.9901475$$

You should notice that each term is smaller than the one before. Because x is a small number the terms containing the higher powers of x become insignificant and we can ignore them.

We say that $[1 + 2(0.01)]^{-1/2}$ is approximately equal to 0.9901475.

Check $[1 + 2(0.01)]^{-1/2}$ using your calculator.

Calculator: [1 + 2 x 0.01] x^y 0.5 +/- = 0.9901475

This answer is the same as the one obtained using the binomial series.

Example 5

Use the binomial series to expand $(1 - 2x)^{1/2}$ to three terms and use this expansion to find the approximate value of $\sqrt{0.96}$ correct to three decimal places.

$$(1 - 2x)^{1/2} = 1 + \frac{(\frac{1}{2})(-2x)}{1!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-2x)^2}{2!} + \ldots\ldots$$

$$= 1 - x - (\tfrac{1}{2})x^2 - \ldots\ldots$$

If $(1 - 2x)^{1/2} = \sqrt{0.96}$, then $1 - 2x = 0.96$

$$2x = 1 - 0.96 = 0.04$$

$$x = 0.02$$

Substitute this value into the expansion.

$$\sqrt{0.96} \approx 1 - 0.02 - \tfrac{1}{2}(0.02)^2 = 1 - 0.02 - 0.0002$$

$$= 0.9798$$

$\sqrt{0.96} =$ 0.980 correct to three decimal places.

Try this next exercise.

Exercise B

1. Expand $(1 - x)^{-1}$ to four terms. Use this expansion to find an approximation for $(1 - x)^{-1}$ when $x = 0.03$.

2. Expand $(1 + {}^{x}/_{2})^{-2}$ to three terms. Use this expansion to find an approximation for $(1 + {}^{x}/_{2})^{-2}$ when $x = 0.02$.

3. Expand $(1 + 2x)^{1/2}$ to four terms. Use this expansion to find an approximation for $(1 + 2x)^{1/2}$ when $x = 0.01$.

4. Expand $(1 + {}^{x}/_{2})^{1/2}$ to three terms. Use this expansion to find an approximate value for $\sqrt{1.03}$ correct to three decimal places.

5. Expand $(1 - {}^{x}/_{3})^{1/2}$ to three terms. Use this expansion to find an approximate value for $\sqrt{0.95}$ correct to three decimal places.

Check your answers with those at the end of the unit.

Application to small errors

The binomial series is given by,

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots\ldots\ldots\ldots$$

We have said that if x is very small, then the terms involving high powers of x are insignificant. We shall now take the case that x is so small that only the first two terms are significant.

In this case,

$$(1+x)^n \approx 1 + \frac{nx}{1!}$$

or,

$$(1+x)^n \approx 1 + nx$$

Consider the following examples.

Example 6

If a metal cube measuring 1m. x 1m. x 1m. is heated it expands to, $(1+x)$m. x $(1+x)$m. x $(1+x)$m.

Find the approximate,

a) increase in area of one face,

b) increase in volume.

a) Original area of one face is $1 \times 1 = 1\text{m}^2$.

New area of one face is $(1+x)$m. x $(1+x)$m. = $(1+x)^2\text{m}^2$.

We now expand $(1+x)^2$ using the binomial series to two terms.

$$(1+x)^n \approx 1 + nx$$

$$(1+x)^2 \approx 1 + 2x$$

The increase in area of one face is approximately equal to,

new area of one face - original area of one face.

$$1 + 2x - 1 = (2x)\text{m}^2.$$

b) Original volume is $1 \times 1 \times 1 = 1\text{m}^3$.

New volume is $(1+x)$m. x $(1+x)$m. x $(1+x)$m. = $(1+x)^3\text{m}^3$.

We now expand $(1+x)^3$ using the binomial series to two terms.

$$(1+x)^n \approx 1 + nx$$

$$(1+x)^3 \approx 1 + 3x$$

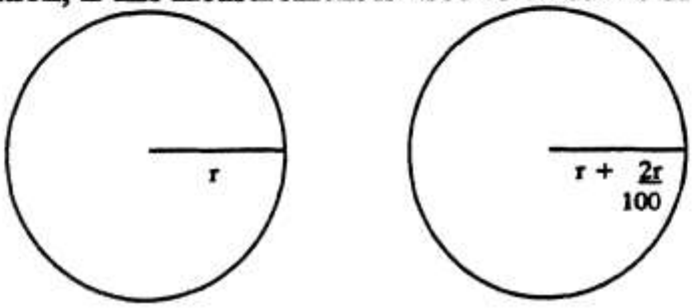
The increase in volume is approximately equal to,

new volume - original volume.

$$1 + 3x - 1 = (3x)\text{m}^3.$$

Example 7

When measuring the radius of a circle, the measurement is 2% too large. Calculate the approximate percentage error resulting using the binomial approximation, if this measurement is used to calculate the area of the circle.



The area of a circle is given by $A = \pi r^2$.

Let the error in the area be δA, we say delta A, then,

$$A + \delta A = \pi[r + (2/100)r]^2$$

$$A + \delta A = \pi r^2(1 + 2/100)^2$$

but $A = \pi r^2$

so $$A + \delta A = A(1 + 2/100)^2$$

We now use the binomial approximation,

$$(1 + x)^n \approx 1 + nx$$

so, $$(1 + 2/100)^2 \approx 1 + 2 \times \frac{2}{100}$$

$$(1 + 2/100)^2 \approx 1 + \frac{4}{100}$$

and $$A + \delta A = A(1 + 4/100)$$

$$A + \delta A = A + (4/100)A$$

$$\delta A = (4/100)A$$

The error is approximately 4% of the actual area.

Example 8

In an experiment to calculate the time period,T, of a pendulum, the length of the string, l was underestimated by 2% and the acceleration due to gravity, g, was overestimated by 3%.

If the time period is given by the formula,

$$T = 2\pi\sqrt{\frac{l}{g}}$$

using the binomial approximation, find the percentage error in the calculation of the time period, T.

$$T + \delta T = 2\pi \frac{[l - ({}^{2}/_{100})l]^{1/2}}{[g + ({}^{3}/_{100})g]^{1/2}}$$

$$T + \delta T = \frac{2\pi l^{1/2}[1 - ({}^{2}/_{100})]^{1/2}}{g^{1/2}[1 + ({}^{3}/_{100})]^{1/2}}$$

$$T + \delta T = \frac{2\pi l^{1/2}}{g^{1/2}}(1 - {}^{2}/_{100})^{1/2}(1 + {}^{3}/_{100})^{-1/2}$$

$$T + \delta T = 2\pi\sqrt{\frac{l}{g}}\ (1 - {}^{2}/_{100})^{1/2}(1 + {}^{3}/_{100})^{-1/2}$$

$$T + \delta T = T(1 - {}^{2}/_{100})^{1/2}(1 + {}^{3}/_{100})^{-1/2}$$

We now use the binomial approximation,

$$(1 + x)^n \approx 1 + nx.$$

$$(1 - {}^{2}/_{100})^{1/2} \approx 1 - ({}^{1}/_{2})({}^{2}/_{100})$$

$$(1 - {}^{2}/_{100})^{1/2} \approx 1 - {}^{1}/_{100}$$

$$(1 + {}^{3}/_{100})^{-1/2} \approx 1 + ({}^{-1}/_{2})({}^{3}/_{100})$$

$$(1 + {}^{3}/_{100})^{-1/2} \approx 1 - {}^{1.5}/_{100}$$

so,

$$T + \delta T \approx T(1 - {}^{1}/_{100})(1 - {}^{1.5}/_{100})$$

$$T + \delta T \approx T(1 - {}^{1.5}/_{100} - {}^{1}/_{100} + {}^{1.5}/_{10000})$$

This is small and considered insignificant.

$$T + \delta T \approx T(1 - {}^{2.5}/_{100}) \approx T - ({}^{2.5}/_{100})T$$

so,

$$\delta T \approx -2.5\% \text{ of } T.$$

There is an underestimate of 2.5% of the time period.

Try the following exercise.

Exercise C

1. Find the percentage error in kinetic energy, E if
$$E = \frac{1}{2}mv^2$$
and the mass, m is overestimated by 3% and the velocity, v is underestimated by 2%.

2. The volume, V of a cone is given by the formula,
$$V = \frac{1}{3}\pi r^2 h$$
where r is the radius and h is the height. Determine the percentage error in the volume if there is an overestimate of 1% in the radius and an overestimate of 4% in the height.

3. The length of the side of a cube is given as l and its volume is V, then
$$l = V^{1/3}.$$
If the volume is underestimated by 6%, what will be the percentage error in the length.

4. The second moment of area, I of a rectangle of length, l and breadth, b about the side of length, b is given by the equation
$$I = \frac{bl^3}{3}.$$
If the length, b is overestimated by 5.5% and the length, l is underestimated by 1.5%, determine the percentage error in the second moment of area, I.

5. The radial acceleration, a during circular motion of radius, r is given by the equation,
$$a = \frac{v^2}{r}$$
where v is the speed of the motion.
Determine the percentage error in a, if v is overestimated by 3.5% and r is underestimated by 2.5%.

Check your answers with those at the end of the unit.

Answers

Exercise A

1. $1 - 2x + 3x^2 - 4x^3 + \ldots\ldots.$
2. $1 - (1/2)x - (1/8)x^2 - (1/16)x^3 - \ldots\ldots.$
3. $1 - 5y + 25y^2 - 125y^3 + \ldots\ldots.$
4. $1 + 6y + 24y^2 + 80y^3 + \ldots\ldots.$
5. $1 - x - (1/2)x^2 - (1/2)x^3 - \ldots\ldots.$
6. $1 + (3/2)x + (27/8)x^2 + (135/16)x^3 + \ldots\ldots.$
7. $1 - 16x + 160x^2 - 1280x^3 + \ldots\ldots.$
8. $1 - 2x + 4x^2 - 8x^3 + \ldots\ldots.$
9. $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots.$
10. $1 - 4x + 12x^2 - 32x^3 + \ldots\ldots.$
11. $1 + (1/2)x - (1/8)x^2 + (1/16)x^3 - \ldots\ldots.$
12. $1 + x + 2x^2 + (14/3)x^3 + \ldots\ldots.$

Exercise B

1. $1 + x + x^2 + x^3 + \ldots\ldots.$
   1.030927
2. $1 - x + (3/4)x^2 - \ldots\ldots.$
   0.9803
3. $1 + x - (1/2)x^2 + (1/2)x^3 - \ldots\ldots.$
   1.0099505
4. $1 + (1/4)x - (1/32)x^2 + \ldots\ldots.$
   1.015
5. $1 - (1/6)x - (1/72)x^2 - \ldots\ldots.$
   0.975

Exercise C

1. The kinetic energy is underestimated by 1%.
2. The volume is overestimated by 6%.
3. The length of the side is underestimated by 2%.
4. The second moment of area is overestimated by 1%.
5. The acceleration is overestimated by 9.5%.