MATHS Xtra
ALGEBRA

Unit 12

Binomial Theorem

Objectives

On completion of this unit you should be able to use:

1. Pascal's triangle.

2. The binomial theorem.

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Pascal's triangle

Consider the following terms.

$3x$ $2a^4$ b^3 $12x^5$

The coefficient of x is 3. The coefficient of $2a^4$ is 2. The coefficient of b^3 is 1. (Remember that b^3 is the same as $1b^3$.) The coefficient of $12x^5$ is 12.

We are going to expand $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$, and $(a+b)^4$ and study the coefficients of the terms obtained.

$(a+b)^0 = 1$

Coefficient 1.

$(a+b)^1 = a+b$

The coefficient of a is 1. The coefficient of b is 1.

Coefficients 1 1.

$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2$
$= a^2 + 2ab + b^2$

The coefficient of a^2 is 1. The coefficient of 2ab is 2.
The coefficient of b^2 is 1.

Coefficients 1 2 1.

$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2 + 2ab + b^2)$
$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$

The coefficient of a^3 is 1. The coefficient of $3a^2b$ is 3.
The coefficient of $3ab^2$ is 3. The coefficient of b^3 is 1.

Coefficients 1 3 3 1.

$(a+b)^4 = (a+b)(a+b)^3 = (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)$
$= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

The coefficient of a^4 is 1. The coefficient of $4a^3b$ is 4.
The coefficient of $6a^2b^2$ is 6. The coefficient of $4ab^3$ is 4.
The coefficient of b^4 is 1.

Coefficients 1 4 6 4 1.

A pattern is emerging and this is called Pascal's triangle.

Pascal's triangle

1

1 1

1 2 1

1 3 -3 1

1 4 6 4 1

Study the pattern.

1 + 1 = 2

1 + 2 = 3 2 + 1 = 3

1 + 3 = 4 3 + 3 = 6 3 + 1 = 4

Study this example.

Example 1

Continue the pattern and use it to expand $(a + b)^5$.

1 4 6 4 1

1 5 10 10 5 1

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Notice that as the powers of a decrease, the powers of b increase.

Try this short exercise.

Exercise A

1. The next row of the pattern is,
   1 6 15 20 15 6 1.
   Use this to expand $(a + b)^6$.
2. Write the next row of the pattern and use it to expand $(a + b)^7$.
3. Write the next row of the pattern and use it to expand $(a + b)^8$.
4. Write the next row of the pattern and use it to expand $(a + b)^9$.
5. Write the next row of the pattern and use it to expand $(a + b)^{10}$.

Check your answers with those at the end of the unit.

Now consider these examples.

Example 2
Expand $(3 + 2x)^4$.

We use,

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

and let a = 3 and b = $2x$

$$\begin{aligned}(3 + 2x)^4 &= 3^4 + 4(3)^3(2x) + 6(3)^2(2x)^2 + 4(3)(2x)^3 + (2x)^4 \\ &= 81 + 216x + 216x^2 + 96x^3 + 16x^4.\end{aligned}$$

Example 3
Expand $(2 - 3x)^3$.

We use,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

and let a = 2 and b = $-3x$

$$\begin{aligned}(2 - 3x)^3 &= 2^3 + 3(2)^2(-3x) + 3(2)(-3x)^2 + (-3x)^3 \\ &= 8 - 36x + 54x^2 - 27x^3.\end{aligned}$$

Try this exercise.

Exercise B
Use Pascal's triangle to expand the following expressions.

1. $(5 + 2x)^3$
2. $(2 + x)^5$
3. $(2 - 3x)^6$
4. $(1 + 2x)^4$
5. $(1 - 2x)^5$
6. $(2 - x)^6$
7. $(2 + 6x)^3$
8. $\left(1 + \frac{x}{2}\right)^3$
9. $(2 - 2x)^5$
10. $(3 - 3x)^6$

Check your answers with those at the end of the unit.

The binomial theorem

You can see from the last exercise that as the powers become higher, it becomes inconvenient to use Pascal's triangle. An alternative to this is the binomial theorem.

The theorem is as follows.

$$(a+b)^n = a^n + \frac{n}{1!}a^{(n-1)}b + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \ldots\ldots\ldots + b^n$$

1! is 1 factorial. This is 1.
2! is 2 factorial. This is 2 x 1 = 2.
3! is 3 factorial. This is 3 x 2 x 1 = 6.

There is a factorial button on your calculator. It has ***x!*** on it.
To find 4!

Calculator: 4 *x!* 24

4! = 24.

Try the following exercise.

Exercise C

Use your calculator to find the value of each of the following.

1.	6!	6.	3!
2.	5!	7.	11!
3.	10!	8.	9!
4.	8!	9.	2!
5.	7!	10.	12!

Check your answers with those at the end of the unit.

Study the examples on the next page.

Example 4
Use the binomial theorem to expand $(2 - 3x)^3$.

This is the question we completed in Example 3 using Pascal's triangle. This time we shall use the binomial theorem.

$$(a + b)^n = a^n + \frac{n}{1!}a^{(n-1)}b + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \dots + b^n$$

Let $a = 2$ and $b = -3x$. n is 3.

$(2 - 3x)^3$

$$= 2^3 + \frac{3}{1!}(2)^2(-3x) + \frac{3(2)}{2!}(2)^1(-3x)^2 + \frac{3(2)(1)}{3!}(2)^0(-3x)^3 + \frac{3(2)(1)(0)}{4!}(2)^{-1}(-3x)^4$$

Note that this last term is zero because one of the terms to be multiplied is zero. Any terms following this would also have zero in and so the series finishes at the x^3 term.

We can now neaten the expression and we obtain
$(2 - 3x)^3 = 8 - 36x + 54x^2 - 27x^3$
as in Example 3.

Example 5
Use the binomial theorem to expand $(x + 2y)^{20}$ to four terms.

$$(a + b)^n = a^n + \frac{n}{1!}a^{(n-1)}b + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \dots\dots\dots + b^n$$

Let $a = x$ and $b = 2y$. $n = 20$.

$$(x + 2y)^{20} = x^{20} + \frac{20}{1!}x^{19}(2y) + \frac{20(19)}{2!}x^{18}(2y)^2 + \frac{20(19)(18)}{3!}x^{17}(2y)^3 + \dots\dots$$

$$(x + 2y)^{20} = x^{20} + 40x^{19}y + 760x^{18}y^2 + 9120x^{17}y^3 + \dots\dots$$

Try the exercise on the next page.

Exercise D

Expand the following using the binomial theorem.

1. $(x + 2y)^2$
2. $(x - 3y)^3$
3. $(2 + 5y)^4$
4. $(3 + 2y)^3$
5. $(4 - 2x)^4$

Expand the following to 4 terms using the binomial theorem.

6. $(2 + 3y)^{10}$
7. $(x - 3y)^{11}$
8. $(2 + 5y)^9$
9. $(-3 + 2y)^5$
10. $(2 - 6x)^7$
11. $(-x + y)^9$
12. $(2x - y)^6$
13. $(-1 + 5x)^{15}$
14. $(3 + 6y)^7$
15. $(-y - 3x)^{10}$

Check your answers with those at the end of the unit.

Now consider the examples on the next two pages.

Example 6

Expand $\left(1 + \frac{x}{2}\right)^6$ to four terms. Use this expansion to find an approximation for 1.05^6.

$$(a+b)^n = a^n + \frac{n}{1!}a^{(n-1)}b + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \ldots + b^n$$

Let $a = 1$ and $b = \frac{x}{2}$. $n = 6$.

$$\left(1 + \frac{x}{2}\right)^6 = (1)^6 + \frac{6}{1!}(1)^5\left(\frac{x}{2}\right) + \frac{6(5)}{2!}(1)^4\left(\frac{x}{2}\right)^2 + \frac{6(5)(4)}{3!}(1)^3\left(\frac{x}{2}\right)^3 + \ldots\ldots.$$

$$\left(1 + \frac{x}{2}\right)^6 = 1 + 3x + \frac{15}{4}x^2 + \frac{5}{2}x^3 + \ldots\ldots.$$

If we now compare $\left(1 + \frac{x}{2}\right)$ to 1.05

then,

$$\frac{x}{2} = 0.05.$$

and $x = 0.1$

We can now substitute this value for x into our binomial expansion.

$$\left(1 + \frac{x}{2}\right)^6 = 1 + 3x + \frac{15}{4}x^2 + \frac{5}{2}x^3 + \ldots\ldots.$$

$$(1 + 0.05)^6 = 1 + 3(0.1) + \frac{15}{4}(0.1)^2 + \frac{5}{2}(0.1)^3 + \ldots\ldots.$$

$$(1.05)^6 = 1 + 0.3 + 0.0375 + 0.0025 \ldots\ldots.$$

$(1.05)^6$ is approximately equal to 1.34.

You can check this using your calculator.

Calculator: 1.05 x^y 6 = 1.3400956

Example 7

Expand $\left(1 - \frac{x}{4}\right)^7$ to four terms. Use this expansion to find an approximation for 0.998^7.

$$(a + b)^n = a^n + \frac{n}{1!}a^{(n-1)}b + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \ldots\ldots\ldots + b^n$$

Let $a = 1$ and $b = -\frac{x}{4}$. $n = 7$.

$$\left(1 - \frac{x}{4}\right)^7 = (1)^7 + \frac{7}{1!}(1)^6\left(\frac{-x}{4}\right) + \frac{7(6)}{2!}(1)^5\left(\frac{-x}{4}\right)^2 + \frac{7(6)(5)}{3!}(1)^4\left(\frac{-x}{4}\right)^3 + \ldots\ldots$$

$$\left(1 - \frac{x}{4}\right)^7 = 1 - \frac{7x}{4} + \frac{21x^2}{16} - \frac{35x^3}{64} + \ldots\ldots$$

If we now compare $\left(1 - \frac{x}{4}\right)$ to 0.998

then,

$$\left(1 - \frac{x}{4}\right) = 0.998$$

and $\frac{x}{4} = 0.002$

$$x = 0.008$$

We can now substitute this value of x into our binomial expansion.

$$\left(1 - \frac{x}{4}\right)^7 = 1 - \frac{7x}{4} + \frac{21x^2}{16} - \frac{35x^3}{64} + \ldots\ldots$$

$$(1 - 0.002)^7 = 1 - \frac{7}{4}(0.008) + \frac{21}{16}(0.008)^2 - \frac{35}{64}(0.008)^3 + \ldots\ldots$$

$$(0.998)^7 = 1 - 0.014 + 0.000084 - 0.00000028$$

$$= 0.98608372$$

You can check this using your calculator.

Calculator: 0.998 x^y 7 = 0.98608372

Now turn over to try the last exercise.

Exercise E

1. Expand $(1 - x)^6$ to four terms. Use this expansion to find an approximation for $(0.990)^6$.

2. Expand $\left(1 - \frac{x}{2}\right)^4$ to three terms. Use this expansion to find an approximation for $(0.992)^4$.

3. Expand $(1 + x)^6$ to three terms. Use this expansion to find an approximation for $(1.06)^6$.

4. Expand $\left(1 - \frac{x}{5}\right)^8$ to three terms. Use this expansion to find an approximation for $(0.995)^8$.

5. Expand $(1 - 2x)^6$ to three terms. Use this expansion to find an approximation for $(0.992)^6$.

6. Expand $\left(1 - \frac{x}{10}\right)^4$ to three terms. Use this expansion to find an approximation for $(0.999)^4$.

7. Expand $(1 + x)^{15}$ to three terms. Use this expansion to find an approximation for $(1.08)^{15}$.

8. Expand $\left(1 - \frac{x}{5}\right)^{20}$ to three terms. Use this expansion to find an approximation for $(0.995)^{20}$.

9. Expand $(2 + x)^5$ to three terms. Use this expansion to find an approximation for $(2.07)^5$.

10. Expand $(3 + x)^6$ to three terms. Use this expansion to find an approximation for $(3.06)^6$.

Check your answers with those at the end of the unit.

Answers

Exercise A

1. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
2. 1 7 21 35 35 21 7 1
   $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
3. 1 8 28 56 70 56 28 8 1
   $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$
4. 1 9 36 84 126 126 84 36 9 1
   $a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$
5. 1 10 45 120 210 252 210 120 45 10 1
   $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$

Exercise B

1. $1(5)^3 + 3(5)^2(2x) + 3(5)(2x)^2 + 1(2x)^3$
   $125 + 150x + 60x^2 + 8x^3$
2. $1(2)^5 + 5(2)^4(x) + 10(2)^3(x)^2 + 10(2)^2(x)^3 + 5(2)(x)^4 + 1(x)^5$
   $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
3. $1(2)^6 + 6(2)^5(-3x) + 15(2)^4(-3x)^2 + 20(2)^3(-3x)^3 + 15(2)^2(-3x)^4 + 6(2)(-3x)^5 + 1(-3x)^6$
   $64 - 576x + 2160x^2 - 4320x^3 + 4860x^4 - 2916x^5 + 729x^6$
4. $1(1)^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + 1(2x)^4$
   $1 + 8x + 24x^2 + 32x^3 + 16x^4$
5. $1(1)^5 + 5(1)^4(-2x) + 10(1)^3(-2x)^2 + 10(1)^2(-2x)^3 + 5(1)(-2x)^4 + 1(-2x)^5$
   $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$
6. $1(2)^6 + 6(2)^5(-x) + 15(2)^4(-x)^2 + 20(2)^3(-x)^3 + 15(2)^2(-x)^4 + 6(2)(-x)^5 + 1(-x)^6$
   $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$
7. $1(2)^3 + 3(2)^2(6x) + 3(2)(6x)^2 + 1(6x)^3$
   $8 + 72x + 216x^2 + 216x^3$
8. $1(1)^3 + 3(1)^2\left(\frac{x}{2}\right) + 3(1)\left(\frac{x}{2}\right)^2 + 1\left(\frac{x}{2}\right)^3$
   $1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}$
9. $1(2)^5 + 5(2)^4(-2x) + 10(2)^3(-2x)^2 + 10(2)^2(-2x)^3 + 5(2)(-2x)^4 + 1(-2x)^5$
   $32 - 160x + 320x^2 - 320x^3 + 160x^4 - 32x^5$
10. $1(3)^6 + 6(3)^5(-3x) + 15(3)^4(-3x)^2 + 20(3)^3(-3x)^3 + 15(3)^2(-3x)^4 + 6(3)(-3x)^5 + 1(-3x)^6$
    $729 - 4374x + 10935x^2 - 14580x^3 + 10935x^4 - 4374x^5 + 729x^6$

Exercise C

1. 720
2. 120
3. 3628800
4. 40320
5. 5040
6. 6
7. 39916800
8. 362880
9. 2
10. 479001600

Answers

Exercise D

The first three answers are given in detail.

1. $(x+2y)^2 = (x)^2 + \frac{2}{1!}(x)^1(2y) + \frac{2(1)}{2!}(x)^0(2y)^2$

   $= x^2 + 4xy + 4y^2$

2. $(x-3y)^3 = (x)^3 + \frac{3}{1!}(x)^2(-3y) + \frac{3(2)}{2!}(x)^1(-3y)^2 + \frac{3(2)(1)}{3!}(x)^0(-3y)^3$

   $= x^3 - 9x^2y + 27xy^2 - 27y^3$

3. $(2+5y)^4 = (2)^4 + \frac{4}{1!}(2)^3(5y) + \frac{4(3)}{2!}(2)^2(5y)^2 + \frac{4(3)(2)}{3!}(2)^1(5y)^3 + \frac{4(3)(2)(1)}{4!}(2)^0(5y)^4$

   $= 16 + 160y + 600y^2 + 1000y^3 + 625y^4$

4. $27 + 54y + 36y^2 + 8y^3$
5. $256 - 512x + 384x^2 - 128x^3 + 16x^4$
6. $1024 + 15360y + 103680y^2 + 414720y^3 \ldots$
7. $x^{11} - 33x^{10}y + 495x^9y^2 - 4455x^8y^3 \ldots$
8. $512 + 11520y + 115200y^2 + 672000y^3 \ldots$
9. $-243 + 810y - 1080y^2 + 720y^3 \ldots$
10. $128 - 2688x + 24192x^2 - 120960x^3 \ldots$
11. $-x^9 + 9x^8y - 36x^7y^2 + 84x^6y^3 \ldots$
12. $64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 \ldots$
13. $-1 + 75x - 2625x^2 + 56875x^3 \ldots$
14. $2187 + 30618y + 183708y^2 + 612360y^3 \ldots$
15. $y^{10} + 30y^9x + 405y^8x^2 + 3240y^7x^3 \ldots$

Exercise E

1.	$1 - 6x + 15x^2 - 20x^3 \ldots$	0.94148
2.	$1 - 2x + 1.5x^2 \ldots$	0.968384
3.	$1 + 6x + 15x^2 \ldots$	1.414
4.	$1 - 1.6x + 1.12x^2 \ldots$	0.9607
5.	$1 - 12x + 60x^2 \ldots$	0.95296
6.	$1 - 0.4x + 0.06x^2 \ldots$	0.996006
7.	$1 + 15x + 105x^2 \ldots$	2.872
8.	$1 - 4x + 7.6x^2 \ldots$	0.90475
9.	$32 + 80x + 80x^2 \ldots$	37.992
10.	$729 + 1458x + 1215x^2 \ldots$	820.854